JOHANNES BRONKHORST

PĀṆINI AND EUCLID: REFLECTIONS ON INDIAN GEOMETRY*

Professor Ingalls – in an article called "The comparison of Indian and Western philosophy" – made the following interesting observation (1954: 4): "In philosophizing the Greeks made as much use as possible of mathematics. The Indians, curiously, failed to do this, curiously because they were good mathematicians. Instead, they made as much use as possible of grammatical theory and argument." This observation should not – as goes without saying in our day and age – be read as a description of the Indian "genius" as opposed to that of the Greeks (at least not in some absolute sense), but as a reminder of the important roles that mathematics and linguistics have played as methodical guidelines in the development of philosophy in Greece and in India respectively. Ingalls appears to have been the first to draw attention to this important distinction.

He was not the last. Ingalls's observation has been further elaborated by J.F. (= Frits) Staal in a few articles (1960, 1963, 1965).[1] Staal focuses the discussion on two historical persons in particular, Euclid and Pāṇini, both of whom – as he maintains – have exerted an important, even formative, influence on developments in their respective cultures. Staal also broadens the horizon by drawing other areas than only philosophy into the picture. To cite his own words (1965: 114 = 1988: 158):

> Historically speaking, Pāṇini's method has occupied a place comparable to that held by Euclid's method in Western thought. Scientific developments have therefore taken different directions in India and in the West. Pāṇini's system produced at an early date such logical distinctions as those between language and metalanguage, theorem and metatheorem, use and mention, which were discovered much later in Europe. In other Indian sciences, e.g., in mathematics and astronomy, as well as in later grammatical systems of Sanskrit, Prakrit, and Tamil, systematic abbreviations are used which not only are ingenuous but also constitute new adaptations of the same method.

The statement "Scientific developments have therefore taken different directions in India and in the West" is of particular interest. It suggests that there are methodical differences between Euclid's *Elements* and Pāṇini's grammar. Staal does not tell us what they are. On the contrary, he emphasizes their common features, as in the following passage (1965: 113 = 1988: 157)

Journal of Indian Philosophy **29:** 43–80, 2001.

When comparing Pāṇini's system with Euclid's *Elements*, a characteristic of the latter, i.e., deduction, appears absent from the former. It is true that there is a kind of deduction in Pāṇini's grammar: *dadhyatra* is deduced from *dadhi atra*, and other forms are similarly deduced with the help of rules. But such deductions do not seem to attain the same degree of generality as Euclid's proofs. However, the difference reflects a distinction of objects, not of structure. In Euclid's geometry, propositions are derived from axioms with the help of logical rules which are accepted as true. In Pāṇini's grammar, linguistic forms are derived from grammatical elements with the help of rules which were framed *ad hoc* (i.e., *sūtras*). Both systems exhibit a structure of logical deduction with the help of rules, and both scholars attempted to arrive at a structural description of facts. In both systems, contradictions and unnecessary complications are avoided. In both cases, the aim is adequate and simple description.

In the immediately following paragraph Staal does mention that there are differences ("More detailed investigations into the methods of Euclid and Pāṇini would throw light on points of difference as well"), but he does not say which ones. Instead he mentions one more common feature: "Another common characteristic is the above-mentioned desire to shorten principles (where Euclid pays attention to minimum number, Pāṇini to minimum length), while disregarding the length of derivations." One looks in vain for a specification of the methodical differences between Euclid and Pāṇini which might explain that "therefore" scientific developments have taken different directions in India and in the West.

Thirty years later Staal still emphasizes the importance of Euclid and Pāṇini, and invokes them as typical examples of the scientific developments in their respective cultures. About Euclid he says (1993: 7): "The ancient Greeks developed logic and a notion of rationality as deduction best exhibited by Euclid's geometry. These discoveries contributed substantially to the development of Western science." About India he observes (1993: 22): "Ancient Indian civilization was an oral tradition and the oral transmission of the tradition became the first object of scientific inquiry. Thus arose two human sciences, closely related to each other in their formal structure: the sciences of ritual and language." The science of ritual, he points out, formulated ordered rules which express the regularities of Vedic ritual. He then continues (pp. 23–24): "The Sanskrit grammarians used rules of precisely this form and demonstrated in a similar manner why many of them have to be ordered. In consequence of metarules, rule-order and other formal properties of rules, Pāṇini developed Sanskrit grammar as a derivational system in some respects more sophisticated than the deductive system of Euclid ..." But when one searches in this later publication for a specification of the methodical differences between Pāṇini and Euclid, one is once again disappointed. Here, too, one finds rather an emphatic statement to the effect that the two are essentially similar (1993: 24):

Science is universal and its main branches developed in all the great civilizations. ... The great traditions of Eurasia are basically similar but there are differences between them in character and emphasis. West Eurasian science is characterized by an emphasis on nature and punctuated increases and decreases in theoretical and empirical sophistication; ... Indian science by formal analysis and an emphasis on human theory.

The "therefore" of "Scientific developments have therefore taken different directions in India and in the West" still remains shrouded in mystery.

The first question that has to be addressed at this point is the following: Is it true that scientific developments have taken different directions in India and in the West? Recall that Ingalls, in the passage cited above, observed that "the Indians, curiously, failed to [make as much use as possible of mathematics in philosophizing], curiously because they were good mathematicians." Indeed, the mathematical literature preserved in India is extensive, though little studied;[2] M.D. Srinivas gives the following estimate (1990: 30): "The recently published 'Source Book of Indian Astronomy'[3] lists about 285 works published in mathematics and mathematical astronomy, of which about 50 are works written prior to the 12th century A.D., about 75 are works written during 12–15 centuries and about 165 are works written during 16–19th centuries." A calculation on the basis of all the volumes of David Pingree's *Census of the Exact Sciences in Sanskrit* (including the one that was not yet accessible to the authors of the 'Source Book of Indian Astronomy') might reveal even larger numbers of works than this.[4] Mathematics, astronomy and medicine[5] were clearly not neglected in classical India. We shall see below that geometry, in any case, was not neglected in Vedic India either.

Should we then drop the thesis of the relative importance of geometry and grammar in Greece and India respectively? Did Pāṇini not exert the influence that Ingalls and Staal attribute to him? Rather than drawing any such abrupt conclusion at his point, I would suggest that the thesis of Ingalls and Staal remains interesting, even though it needs, and deserves, closer attention, and may have to be tested against more evidence than it has been exposed to so far. It is an undeniable fact that many, if not all, classical Sanskrit authors had studied Sanskrit grammar and therefore undergone the influence of Pāṇini and his early commentators. The extent to which this omnipresent influence affected philosophy and the sciences remains to be determined, but there is no dispute over the fact that such an influence could have taken place, if for no other reason than that the authors concerned had spent time studying grammar.

One rather obvious question has rarely been asked, certainly not by Ingalls and Staal, viz. to what extent grammatical influence is noticeable in Indian mathematical literature, and the extent to which Pāṇini's work may have provided a methodical guideline for the authors of mathematical texts.[6] This question will gain in importance if we concentrate on the portions on geometry in Indian mathematical texts, because here we will be able to make a direct comparison with Euclid. If we find obvious differences in method between Euclid and classical Indian geometry, we may then ask whether and to what extent these differences are due to the influence of Pāṇini.[7] The fact that the geometry of Indian mathematical astronomy may ultimately derive from Euclidean geometry will make this question all the more poignant.[8]

Let me cite a second time the following passage from Staal's article "Euclid and Pāṇini": "When comparing Pāṇini's system with Euclid's *Elements*, a characteristic of the latter, i.e., deduction, appears absent from the former. It is true that there is a kind of deduction in Pāṇini's grammar: *dadhyatra* is deduced from *dadhi atra*, and other forms are similarly deduced with the help of rules. But such deductions do not seem to attain the same degree of generality as Euclid's proofs. However, the difference reflects a distinction of objects, not of structure."[9] This statement claims that the fact that deductions of a certain type are present in Euclid and absent in Pāṇini is due to the distinction between the objects which these two sciences are dealing with. It strongly suggests that a competent Indian dealing with geometry rather than with grammar would use the same kind of deductions as Euclid.

This, of course, is a testable suggestion, and the remainder of this article is meant to test it against the evidence provided by Bhāskara's commentary on Āryabhaṭa's Āryabhaṭīya, one of the earliest texts that provides us first-hand information about classical Indian geometry in practice. Before however doing so, it will be useful to specify somewhat more precisely what the methodical influence of Pāṇini's would look like. What are the principal characteristics of Pāṇini's grammar?

This question was already raised in the Mahābhāṣya of Patañjali, a grammatical work dating from the second century B.C.E. whose study accompanied that of Pāṇini's grammar throughout most of the classical period. That is to say, students who had to acquaint themselves with Pāṇini's grammar also had to learn the analysis of its method that is

presented in the Mahābhāṣya. This analysis occurs in the introductory chapter of the Mahābhāṣya (the Paspaśāhnika), no doubt the best known portion of this work. Starting from the position that grammar should teach words, Patañjali then raises the question as to how this is to be done. There are far too many of them to be enumerated one by one. Patañjali then continues:[10] Words must be learned by means of general principles and specific features, laid down in general rules and exceptions. This is indeed how Pāṇini's grammar works: rules are formulated that are as general as is possible; where necessary exceptions are added to make sure that the general rule will not be applied where this is not desired. The greatest possible generality, in combination with extreme concision of expression, constitute the two most important characteristics of Pāṇini's grammar.

With this in mind we can turn to the mathematical texts. The Āryabhaṭīya of Āryabhaṭa (= Āryabhaṭa I) dates from 499 (Pingree) or 510 (Billard) C.E., and contains a chapter on mathematics entitled Gaṇitapāda which is "the earliest text of this genre that we have" (Pingree, 1981: 56).[11] The Āryabhaṭīya is written in *āryā* verses which mainly contain rules formulated in a highly condensed form. The rules of the Gaṇitapāda in part concern arithmetic, in part geometry. Indeed, the commentator Bhāskara (to be introduced below) clearly states that mathematics (*gaṇita*) is twofold: arithmetic and geometry, or perhaps more precisely: arithmetic of quantities and arithmetic of geometrical figures.[12]

From among Āryabhaṭa's rules, let us concentrate on the probably best known theorem of Euclidean geometry, viz. the Pythagorean theorem. This theorem finds expression in the first half of verse 17 of the Gaṇitapāda, which reads:[13] "The square of the base [of a right-angled trilateral] and the square of [its] upright side is the square of the hypotenuse." No more is said about it: no proof is given, no examples and no diagrams. We have seen that Staal looks upon the concision of Indian mathematical texts as an adaptation of Pāṇini's method. Others maintain that this conciseness is a sign of the scientific character of Āryabhaṭa's text, as does Pierre-Sylvain Filliozat in the following passage:[14]

Cette forme littéraire, en particulier le recours à l'ellipse de l'expression et de la pensée, a une autre raison d'être, en plus de l'obéissance à la convention culturelle d'un pays et d'une époque. Elle a une utilité de caractère scientifique, en ce qu'elle relève de la formalisation. Les intellectuels indiens ont reconnu très tôt l'aide apportée à la pensée par l'économie de l'expression, l'outil efficace qu'est un langage technique abrégé. Le premier exemple de formalisation scientifique dans l'histoire universelle des sciences semble bien être le formulaire de linguistique qu'est le *sūtra* de Pāṇini et qui se caractérise par l'emploi d'une métalangue et le passage obligé au niveau de la plus grande généralité afin de permettre l'application d'une formule unique au plus grand nombre possible de cas particuliers. Āryabhaṭa n'a pas poussé la formalisation aussi loin que le grammairien. Mais il ne l'a pas ignorée. Il n'a pas de métalangue à proprement parler, mais il a son vocabulaire technique et surtout il vise le plus haut degré de généralité dans ses propositions. Ceci est une des valeurs scientifiques de son oeuvre.

Filliozat is right in that the Pythagorean theorem is presented in its most general form. Instead of enumerating a number of values valid for specific rectangular triangles (3: 4: 5; 5: 12: 13; etc.) Āryabhaṭa formulates this rule for all possible rectangular triangles, and for all possible values of its sides. One might consider the possibility that the very (re-)discovery of, and interest for, the Pythagorean theorem (and of other theorems) by Indian mathematicians is due to the obligation to formulate their knowledge in as concise, and therefore as general, a form as possible.[15] But it must be added that the Gaṇitapāda, perhaps because of this same "Pāṇinian" conciseness, presents the Pythagorean theorem (and many other theorems, as we will see) without a proof, and therefore without a hint as to why we should accept it. One might be tempted to conclude from this that the search for brevity, which turned out to be such a great advantage for grammar, was a mixed blessing for mathematics. We will see below that such a conclusion would not do full justice to all the available evidence.

[The part of Brahmagupta's Brāhmasphuṭasiddhānta (628) dedicated to geometry (12.21–47) is as concise as Āryabhaṭa's text, and much the same could be said about it. The Pythagorean theorem is formulated as follows:[16] "Subtracting the square of the upright from the square of the diagonal, the square-root of the remainder is the side; or subtracting the square of the side, the root of the remainder is the upright: the root of the sum of the squares of the upright and side is the diagonal." Here too, no proof, no examples, no diagrams.

A third text on mathematics which may perhaps date from a period close to Brahmagupta (Hayashi, 1995: 149), the Bakhshālī Manuscript, contains in its preserved portions little about geometry (see Hayashi, 1995: 97): essentially only the sūtra which Hayashi calls N14 (p. 401; Sanskrit pp. 221 ff.; English translation pp. 322 ff.). This sūtra, which prescribes a way to calculate the volume of a particular solid, is accompanied by an example. The fragmentary state of this text does not allow us to draw far-reaching conclusions.]

It is not altogether fair to judge highly condensed fundamental texts from classical India without taking into consideration that such texts were always accompanied by one or more commentaries, whether oral or written. This is true for Pāṇini's Aṣṭādhyāyī, which was the starting point for an extensive commentatorial tradition, and which must have been accompanied by some kind of commentary from the very beginning. Patañjali's Mahābhāṣya – which is a commentary on the Aṣṭādhyāyī, even though a special kind of commentary – was composed well before the beginning of the common era. More straightforward commentaries on the Aṣṭādhyāyī no doubt existed from an early time onward, and even if the oldest surviving commentary of this nature – the Kāśikā Vṛtti – is a relatively late text (around 700 C.E.), it is known that earlier such commentaries existed.[17] Āryabhaṭa's text, too, may initially have been studied under the guidance of a teacher, who would orally provide many of the elements that are lacking in the verses. Written commentaries were subsequently composed, the earliest ones of which have not survived. The earliest mathematical commentary that *has* survived[18] is the Āryabhaṭīya Bhāṣya[19] of Bhāskara (= Bhāskara I), which dates from 629 C.E.,[20] i.e. almost exactly from the same year as Brahmagupta's Brāhmasphuṭasiddhānta. By contrast, the earliest surviving commentary on Brahmagupta's Brāhmasphuṭasiddhānta – Pṛthūdakasvāmin's Vivaraṇa – dates from 864, and belongs to the later period of Śrīdhara and others which will not be considered here.

It is in Bhāskara's Āryabhaṭīya Bhāṣya, then, that we will look for the elements that are so obviously missing from Āryabhaṭa's Gaṇitapāda – assuming, for the sake of argument, that Staal's assessment of Indian geometry is correct.[21] Three of these elements have already been mentioned. What is primarily missing in the fundamental text are a proof of the theorem concerned, examples of how and where it can be applied, and where necessary one or more geometrical diagrams that facilitate visualizing the theorem and its proof; in view of the Indian commentatorial customs in other areas we may also expect to find explanations and (re-)interpretations of the exact formulation of the basic text, where necessary.

With this fourfold expectation in mind we turn to the Āryabhaṭīya Bhāṣya on Gaṇitapāda 17ab. We do indeed find here an explanation of the words of the basic text (which in this case is very short, given the unproblematic formulation of the theorem),[22] followed by examples (tasks with solutions, all of which concern triangles the lengths of whose sides relate to each other as 3: 4: 5) and diagrams. The only thing that is lacking is a proof of the theorem!

For a reader schooled in Euclidean geometry this is very surprising. An important, nay fundamental, theorem of geometry is presented, explained and illustrated by Bhāskara without proof![23] This absence of proof does not stop Bhāskara from frequently using this theorem elsewhere in his commentary. He cites it – in the form given to it by Āryabhaṭa in verse 17ab – in his comments on Gaṇitapāda 6ab (p. 56 l. 5), 6cd (p. 58 ll. 14–15), 17cd (p. 103 ll. 12–13), and uses it implicitly at many other occasions. The theorem is already implied in the explanation-cum-etymology which Bhāskara gives of the difficult term *karaṇī* "surd" in his introductory remarks to verse 1:[24] "The *karaṇī* is [so called] because it makes (*karoti*) the equation of the diagonal (c) and the sides (a and b) [of a rectangle] ($a^2 + b^2 = c^2$)."

It goes without saying that the absence of a proof in the case of the Pythagorean theorem raises the question whether any proofs are presented in connection with other theorems. This question has been investigated by Takao Hayashi in the Introduction of his book *The Bakhshālī Manuscript* (1995), where he studies in particular the use of the terms *pratyaya(-karaṇa)* and *upapatti*. We will consider some of the passages concerned below, but note here already that Hayashi arrives at the following conclusion (p. 76): "As a matter of fact ... we can hardly find proofs or derivations of mathematical rules in Bhāskara's commentary on the Āryabhaṭīya."

To illustrate this point, we shall consider a passage which might at first sight create the impression that it does provide – or rather: hint at – a proof of a rule; we will see that it doesn't. The rule concerned is presented in the second half of verse 17: "In a circle the product of the two sagittas of the two arcs [that together constitute the circle] equals the square of half the chord."[25]

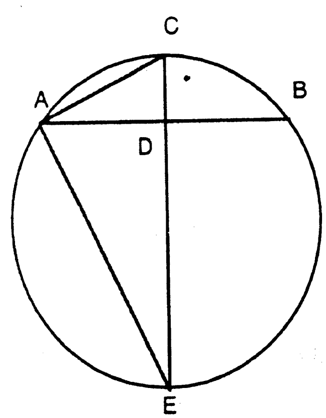

Figure 1. CD × DE = AD2.

That is to say: $CD \times DE = AD^2$. This rule can easily be proved with the help of the Pythagorean theorem:

$$AC^2 = AD^2 + CD^2$$
$$AE^2 = AD^2 + DE^2$$
$$CE^2 = (CD + DE)^2 = AC^2 + AE^2 = AD^2 + CD^2 + AD^2 + DE^2$$
$$CD^2 + DE^2 + 2 \times CD \times DE = CD^2 + DE^2 + 2 \times AD^2$$
$$CD \times DE = AD^2 \quad \text{q.e.d.}$$

Bhāskara does not however do so, at least not at the beginning of his comments on this rule. Instead he gives a large variety of examples (tasks with solutions), which cover more than six pages of the edition. Having provided (numerical) solutions to all the problems Bhāskara then states:[26] "The verification (*pratyayakaraṇa*) in the case of all these surfaces [is obtained] with the [rule according to which] 'the square of the hypotenuse equals the sum of the squares of the two rectangular sides'." In spite of the impression which this statement might create, this is not a reference to the proof of the theorem, but a reminder that the Pythagorean theorem can be used, instead of the present theorem, to carry out the calculations and find the solutions to the problems discussed. That is to say, the Pythagorean theorem can be used to verify the solutions that had been reached with the help of the other theorem. This may be a justification of sorts of that theorem, but hardly a proof in any of its usual senses.[27]

If we follow Staal in thinking that the study of geometry should more or less automatically lead to deductions of the Euclidean kind, then the absence of proofs in Bhāskara's commentary demands an explanation. It might then be symptomatic of a fundamental difference between Indian and Greek geometry. This difference in its turn, still according to Staal, might be due to the methodic influences of Pāṇini and Euclid respectively. Is it conceivable that Pāṇini's grammar may have had this effect on classical Indian geometry? Let us consider some aspects of this question in more detail.[28]

Pāṇini's grammar, as was pointed out above, had been studied by all or almost all classical Brahmanical scholars in whatever disciplines, as well as by others (such as Buddhist and Jaina scholars). It is almost superfluous to recall that Bhāskara's commentary contains ample signs that its author, too, was thoroughly acquainted with it (even though his text, in its edited form, contains a number of grammatical infelicities; see note 59 below). The fourth appendix to K.S. Shukla's edition of the text enumerates eight quotations from Pāṇini's Aṣṭādhyāyī, one from its

Dhātupātha, ten from Patañjali's Mahābhāṣya,[29] one from Bhartṛhari's Vākyapadīya,[30] and one passage referring to the list of sounds that precedes Pāṇini' s grammar (the "Śivasūtras"). This enumeration is however far from complete.[31] A quotation from the Mahābhāṣya that has not been identified by the editor is the line *sāmānyacodanāś ca viśeṣe 'vatiṣṭhante* (p. 39 l. 23; p. 55 ll. 12–13), which is cited from the Mahābhāṣya on Pāṇ. 4.1.92 vt. 3.[32] Other quotations that could be added to the list are the following. The comparison *yathā 'kṅiti ca' ity atra luptanirdiṣṭo gakāraḥ* (p. 9 ll. 18–19) refers, without saying so, to a discussion in the Mahābhāṣya on Pāṇ. 3.2.139. The line *yaś ca sarvaś ciraṃ jīvati sa varṣaśataṃ jīvati* (p. 13 l. 16) echoes a line from the Paspaśāhnika;[33] *dṛṣṭānuvidhitvāc chandasaḥ* (p. 14 l. 22) is part of Pāṇ. 1.1.6 vt. 1; *iṅgitena ceṣṭitena nimiṣitena mahatā vā sūtraprabandhena [ca] ācāryāṇām abhiprāyo gamyate* (p. 34 ll. 16–17) is quoted from the Mahābhāṣya on Pāṇ. 6.1.37;[34] *tātsthyāt tācchabdyam* (not °*chābdyam*; p. 35 l. 10) occurs on Pāṇ. 5.4.50 vt. 3.[35] Particularly interesting is the line *anityaḥ samāsāntavidhiḥ* (p. 23 l. 22); this reflects a position taken in the Mahābhāṣya on Pāṇ. 6.2.197 (III p. 140 l. 6: *vibhāṣā samāsānto bhavati*) but is identical in form with a grammatical paribhāṣā which we find, perhaps for the first time, in Vyāḍi's Paribhāṣāvṛtti (no. 69: *samāsāntavidhir anityaḥ*; Wujastyk, 1993: I: 70).[36] Equally interesting is Bhāskara's use of the grammatical expression *sambandhalakṣaṇā ṣaṣṭhī* (p. 38 l. 2), which does not appear to occur in the Mahābhāṣya, but which is attested in the Kāśikāvṛtti (on Pāṇ. 7.1.90). The editor has furthermore not identified *a i uṇ ṛ ḷk* (p. 18 l. 16), which is "Śivasūtra" 1 & 2. To this must be added that there are many implied references to sūtras of Pāṇini. One occurs on p. 60 ll. 8–9 of the edition, where the correct reading must be *anyapadārthena* (rather than *anyapādārthena*) *samapariṇāhaśabdena kṣetrābhidhānāt*, which means: "because the expression *samapariṇāha* – [being a *bahuvrīhi* compound and therefore] referring to something else [on account of Pāṇ. 2.2.24: *anekam anyapadārthe* (*bahuvrīhiḥ*)] – would denote the field". Similarly, p. 116 ll. 2–3: *trirāśiḥ prayojanam asya gaṇitasyeti trairāśikaḥ* "the *trairāśika* ('rule of three') is called thus because three quantities (*trirāśi*) are the purpose (*prayojana*) of this calculation" contains an implicit reference to Pāṇ. 5.1.109 *prayojanam*, which prescribes the suffix *ṭhaÑ* (= *ika*). The immediately preceding line *trayo rāśayaḥ samāhṛtāḥ trirāśiḥ* "three quantities combined are [called] *trirāśi*" implicitly refers to Pāṇ. 2.1.51 (*taddhitārthottarapadasamāhāre ca*) and 52 (*saṅkhyāpūrvo dviguḥ*) which prescribe the compound called *samāhāra dvigu*, which is a kind of *tatpuruṣa*. P. 7 l. 3 *tasmin krakace bhavaḥ krākacikaḥ* and p. 11 l. 3 *navānte bhavaṃ navāntyam* similarly refer to Pāṇ. 4.3.53 *tatra bhavaḥ*; p. 180 l. 15 *ādau bhavati iti ādyaḥ* to Pāṇ. 4.3.54 (*digādibhyo yat*). P. 2 l. 4 *ktvāpratyayena pūrvakālakriyā 'bhidhīyate* evokes Pāṇ. 3.4.21 *samānakartṛkayoḥ pūrvakāle* (*ktvā 18*). In a way each and every analysis of a compound – of which there are numerous instances in the Āryabhaṭīya Bhāṣya, even though their technical names are rarely used – can be considered an implicit reference to Pāṇini's grammar. The influence of Patañjali's Mahābhāṣya on early mathematical literature can also be deduced from the fact that Bhāskara refers to the works of Maskari(n), Pūraṇa and Mudgala as providing the *lakṣaṇa* and the *lakṣya* of various branches of mathematics; these are precisely the terms used in the Paspaśāhnika of the Mahābhāṣya to designate grammar: *lakṣaṇa* designates the sūtras, *lakṣya* the objects to be studied, i.e. words in the case of grammar.[37] It is in this context interesting to note that Bhāskara refers to Āryabhaṭa's text using these same expressions *lakṣaṇa* and *sūtra*: *lakṣaṇa* infrequently, e.g. p. 74 l. 3; *sūtra* very often.[38] This means that Bhāskara, like Patañjali, uses the terms *lakṣaṇa* and *sūtra* as synonyms. This is particularly clear on p. 49 l. 16, where the two are simply juxtaposed to each other; p. 51 l. 12 cites a *lakṣaṇasūtra*. [It is clear from his discussion that these sūtras do not always coincide with the verses of Āryabhaṭa,[39] but can be parts of them. An extreme case is verse 19 of the Gaṇitapāda, in which Bhāskara distinguishes no fewer than five different sūtras.[40]]

The presentation of geometrical theorems without proof raises the question whether perhaps proofs were known to the authors concerned, but were not recorded in their texts because their literary model – Pāṇini's grammar and its commentaries, as well as other texts that had adopted this model – did not really leave place for them. It is at least conceivable that Āryabhaṭa and Bhāskara (or the mathematicians from whom they took their material) practiced geometry on a more abstract level than is evident from their writings, making sure that the theorems they presented to their readers were based on solid proofs.[41] One would be free to think so, were it not that these two authors present some theorems that are quite simply wrong.[42] This cannot but mean that these theorems were not based on solid proofs, for no such proofs are possible. (It does not exclude the possibility that some form of verification existed; this system of verification, if it existed, was not however capable of discovering and correcting these errors.)

Āryabhaṭa is wrong where he gives the volume of a pyramid as:[43] "Half the product of the height and the [surface of the triangular base] is the volume called 'pyramid'." The correct volume of a pyramid is a third, not half, of the product here specified. In spite of this, Bhāskara accepts Āryabhaṭa's rule and carries out some (incorrect) calculations with its help. The same is true of Āryabhaṭa's incorrect rule for the volume of a sphere.[44]

Āryabhaṭa's incorrect rules have drawn the attention of scholars, some of whom have tried to interpret the rules differently, so as to obtain a correct result.[45] Filliozat and Mazars object to such a reinterpretation in a joint publication (1985), pointing out that Bhāskara's comments do not support it.[46] They further make the following important observation (p. 40): "Il faut comprendre Āryabhaṭa, non pas dans ses seuls résultats transposés en formules modernes, mais dans sa propre culture intellectuelle." In a more recent publication, Filliozat states (1995: 39): "It is an error to actualise the ideas of past scientists, to transpose in modern terminology their ancient expressions."

Filliozat and Mazars are no doubt right in this, and one would expect that they might try to explain Āryabhaṭa's errors in the light of "sa propre culture intellectuelle". Unfortunately they don't do so. These errors remain a total mystery to them, as is clear from the concluding words of their article (p. 46):

> L'existence de telles erreurs chez le grand mathématicien est certes surprenante. Elle l'est encore plus chez Bhāskara, puisque nous savons que son contemporain Brahmagupta connaissait, au moins pour le volume de la pyramide, la formule exacte.

And yet it seems obvious that these mistakes do throw light on the intellectual culture of their authors. It seems inevitable to conclude that the theorems propounded by Āryabhaṭa and Bhāskara were apparently not accompanied by proofs, not even in private, not even outside the realm of the written commentary.[47] They were handed down as received truths, with the result that incorrect theorems were not identified as a matter of routine by any student who checked the proofs.[48] Bhāskara confirms that he regards the theorems and other information contained in Āryabhaṭa's work as received truths in his commentary on the first chapter, the Gītikāpāda. Here he states, under verse 2, that all knowledge derives from Brahmā; Āryabhaṭa pleased Brahmā on account of his great ascetic practices and could then compose, for the well-being of the world, the ten Gītikāsūtras on planetary movement (chapter 1), as well as the one hundred and eight *āryā* verses on arithmetic (ch. 2), time (ch. 3) and the celestial globe (ch. 4).[49] Elsewhere (p. 189 ll. 14–15) he calls Āryabhaṭa *atīndriyārthadarśin* "seeing things that are beyond the reach of the senses". It is further noteworthy that in the third

chapter, the Kālakriyāpāda, Bhāskara cites, under verse 5, a verse from an earlier (non-identified) astronomical text – about the months in which the sun passes the summer and winter solstices – and calls it Smṛti, more precisely: *our* Smṛti (p. 182 l. 9: *asmākaṃ smṛtiḥ*); he bolsters this position with an argument based on Mīmāṃsā, unfortunately not completely clear in all its details, but which in outline looks as follows: this Smṛti has to be accepted, for it is neither in conflict with the Śruti, nor with perception. Calling a text Smṛti and insisting that it conflicts neither with the Śruti nor with perception amounts to granting it the same authority as the Veda itself, and maintaining that it cannot be wrong.

It seems clear from what precedes that there is little space in Bhāskara's geometry for proofs, and none at all for definitions and axioms, the starting points of perfect proofs. This raises the following fundamental question. If the geometrical figures dealt with by the Indian mathematicians under consideration are not constituted of elements laid down in definitions,[50] what then is the geometry of Āryabhaṭa and Bhāskara *about*? If its triangles, circles, pyramids and spheres are not abstractions, what then are they?

We may find a clue to the answer in the following passage of Bhāskara's Bhāṣya on Gaṇitapāda verse 11:[51]

> We, on the other hand, maintain that there is a chord equal to the arc [which it subtends]. If there were no chord equal to the arc [it subtends], an iron ball would not be stable on level ground. We infer from this that there is an area such that the iron ball rests by means of it on the level ground. And that area is the ninety-sixth part of the circumference.
>
> Other teachers, too, have accepted a chord equal to the arc [which it subtends, as is clear from] the following quotation: 'From the body of a sphere, a hundredth part of its circumference touches the earth'.[52]

Both Bhāskara's own words and the line he cites seem to confuse the surface of a sphere and the circumference of a circle. This may for the moment be looked upon as a detail.[53] The most important thing to be learnt from this passage is that for Bhāskara – and no doubt for other teachers as well – spheres and circles are no (or not only) abstract objects, but (also) concrete things whose features are at least to some extent determined by their behavior in the world of our daily experience. A sphere, we learn, has to have flat surfaces, for only thus can it be stable on level ground.

The question of the absence of proofs in Indian geometry acquires a different dimension once it is clear that Euclidean and classical Indian geometry may not really concern the same objects. The objects of classical Indian geometry – its triangles, circles, spheres etc. –

are no mere abstractions, but things present in the outside world. (The fact that the use of triangles formed by such imprecise things as shadows and flames inevitably implies that the mathematician worked with idealized objects, does not necessarily contradict this.) As such they resemble the objects of grammatical analysis far more than do Euclidean diagrams. The objects of grammar – sounds, words, phrases – are there in the outside world, and grammar is therefore a science which deals with objects whose existence is quite independent of the volition of the grammarian. To invoke Patañjali's words: one does not visit a grammarian when in need of words as one visits a potter when one needs a jar.[54] In other words, the objects of grammar, unlike pots, are not fashioned in accordance with the wishes of their users. Elsewhere Patañjali states:[55] "Words are authoritative for us; what the word says is our authority." Once again, words do not adjust to grammarians, but grammarians follow the dictates of words. Grammar presents multiple linguistic data with the help of rules that are as general as possible, but that cannot go against the data of language. Similarly, the rules of classical Indian geometry describe numerous geometrical forms that exist in the outside world with the help of rules that are as general as possible. The fact that these rules describe concrete objects rather than mathematical abstractions has as consequence that they may often be generalizations from concrete observations, rather than general statements derived from first principles.[56] The Pythagorean theorem, seen this way, may therefore in the classical Indian context be a generalization of measurements taken from various rectangular triangles; the notion of a general proof for this theorem might have appeared to the early Indian mathematicians as completely out of place, just as to a grammarian a general proof of the grammatical rule that *i* followed by *a* is replaced by *y*, not only in *dadhyatra* out of *dadhi atra* but also in all other similar situations, would have looked strange.

Having said this, it should not of course be overlooked that geometrical figures and bodies are not the same kinds of things as linguistic expressions. Numerous geometrical rules become obvious with the help of minimal manipulation of figures. Take the rule that the area of a triangle is half the product of its height and its base.[57] This rule follows almost automatically from a look at the following diagram, and there is no need to assume that it is a generalization of a number of measurements.[58] And yet the notion of general proof does not characterize Bhāskara's discussions of geometry.

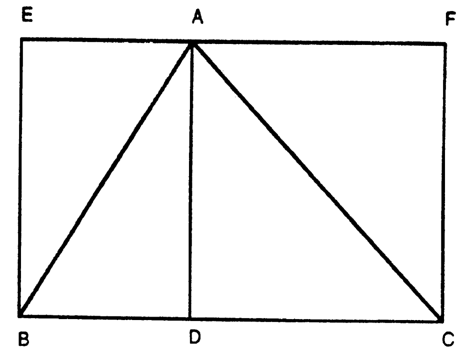

Figure 2. The area of triangle ABC is half the area of rectangle EFCB.

A passage in Bhāskara's comments on Gaṇitapāda verse 10 is interesting in this context. We read here:[59]

[Question:] Why is the approximate circumference [of a circle] given, not its exact circumference?
[Response:] People think as follows: there is no way in which the exact circumference [can] be calculated.
[Question:] Isn't there the following saying: the circumference of a circle is the square root of ten times the square of the diameter?[60]
[Response:] Here too, [the claim] that the circumference of a diameter that equals one is the square root of ten is mere tradition, and no demonstration (*upapatti*).
[Objection:] And if one thinks that the circumference, as directly (*pratyakṣeṇa*) measured, of a [circular] area whose diameter equals one is the square root of ten,
[Response:] this is not [correct], for the measure of surds cannot be expressed.
[Objection:] If [it is maintained] that the circumference of [a circle] having that diameter (viz. one), surrounded by the diagonal of an oblong quadrilateral whose width and length are one and three [respectively], has the [same] size [as that diagonal],
[Response:] that, too, has to be established.

This passage is interesting for various reasons. For one thing, it uses the expression *upapatti*, which some translate "proof", but for which "demonstration" seems more appropriate. It also states quite unambiguously that "the measure of surds cannot be expressed [in fractions (?)]".[61] But its special interest for us at present lies in the fact that it mentions, and accepts in principle, the idea of direct measurement as a method to reach conclusions of a geometrical nature. It even indicates how the square root of ten can be measured, so as to compare its length with the circumference of a circle. It appears to be legitimate for Bhāskara to measure the circumference of a circle, presumably with a cord, and derive from this measurement what we would call the value of π. All that he demurs to is the claim that the result of this measurement – in the case of a circle whose diameter is one – will be the square root of ten. This cannot be true, because the square root of ten, like other surds, cannot be expressed, presumably in terms of units or fractions.

It is hard to know what "demonstration" would have convinced Bhāskara of the truth of the claim that $\pi = \sqrt{10}$. But one thing is sure. The only other time he uses the term *upapatti* in this same commentary, it does not refer to anything like what we would call a proof.[62] The passage concerned occurs in the middle of a discussion of an example under Gaṇitapāda verse 6cd. Hayashi (1995: 75; modified) translates it as follows:[63] "Drawing a plane figure in order to show the ground (*upapatti*) of the following Rule of Three".

Bhāskara goes on criticizing the claim that $\pi = \sqrt{10}$ at great length. Part of his criticism is interesting because it reveals that he appears to have extended the respect which he felt for the rules contained in the Āryabhaṭīya to other rules, which he perhaps found in other treatises. One way in which he tries to demonstrate the insufficiency of this value for π is by showing that it leads to totally unacceptable consequences. The following passage illustrates this:[64]

And the calculation of an arc on the basis of the assumption that the measurement of a circumference is made with the square root of ten, is not always [possible]. For the sūtra for calculating an arc is [the following] half *āryā* verse:

The sum of a quarter of the chord and half the sagitta, multiplied by itself, ten times that, the square root of that.

[Consider] now the following example: In [a circle] whose diameter is fifty-two, the length of the sagitta is two.

With the help of the rule "*ogāhūṇaṃ vikkhaṃbham*" one obtains as length of the chord: twenty (20). With this chord the calculation of the arc becomes: a quarter of the chord is 5, half the sagitta is 1, their sum is 6, multiplied by itself: 36, ten times that: 360, the square root of that is the arc.

The square of the whole chord is four hundred, the square of the arc three hundred sixty; how is that possible? The arc must [certainly] be longer than the chord. Here [on the other hand] the arc being thought out by [these] extremely clever thinkers has turned out to be shorter than the chord! For this reason, homage be paid to this root of ten, charming but not thought out.

This passage cites two rules, both of which Bhāskara calls sūtras. The second of these is "*ogāhūṇaṃ vikkhaṃbham*", which had been cited in full, and illustrated, on the preceding page (p. 73 l. 2 ff.); it is in Prakrit and no doubt derives from a Jaina text.[65] Contentwise this rule is no different from verse 17cd of the Gaṇitapāda, studied above, and it is surprising that Bhāskara does not mind citing (and obviously accepts without questioning) the rule in the form in which it was used by the Jainas. The first sūtra cited in this passage, on the other hand, is in

Sanskrit; it is not known where it comes from, but clearly it offers a calculation involving the square root of ten. Bhāskara uses this rule, and shows that it leads to an absurd outcome. Strictly speaking this may be due (i) to the particular value assigned to π; (ii) to the form of the rule in general, quite apart from the value assigned to π; or (iii) to both of these at the same time. Bhāskara triumphantly concludes that the fault lies with the square root of ten. A small calculation would have shown him that the approximate value of π which he, following Āryabhaṭa's Gaṇitapāda verse 10, does accept ($\pi = 3.1416$), if used with the same but adjusted formula, leads to the same absurdity ($6 \times 3.1416 = 18.8496$ for the length of arc; 20 for the chord). We must conclude that – unless Bhāskara had lost his mathematician's mind while writing this passage – he really criticized the formula, which may have occurred in a presumably Jaina treatise.[66]

This allows us to draw some further conclusions. Bhāskara might, from a modern point of view, have criticized the wrong formula for calculating a length of arc by contrasting it with a correct one, along with a proof for the latter.[67] He did not do so, most certainly because he did not have a correct formula, much less a proof for it.[68] What is more, it is highly unlikely that he thought in terms of proofs of correct formulae. From his point of view the rules and theorems he had inherited were correct, the ones others had inherited were presumably mostly correct, but some of these led to unacceptable results, and were therefore incorrect.

The preceding reflections teach us the following. Judging by the evidence discussed so far, classical Indian geometry and grammar share a number of features, which are compatible with (but do not prove) the assumption that Pāṇini's grammar did indeed exert an influence on the former. Two features in particular deserve mention:

1. Classical Indian geometry, like grammar, describes objects that exist in the material world, not abstractions. The practice of geometry does not therefore exclude the physical manipulation of such objects, and the search for generalizations based on concrete measurements. This explains that some conclusions – such as the not totally spherical shape of a sphere – may be based on reflections about or observations of material objects.
2. The objects of both classical Indian geometry and Sanskrit grammar are described with the help of rules that are as general as possible. In the case of geometry, the resulting rules look like Euclidean theorems, but unlike the latter, and like the rules of Pāṇini,

they do not derive their validity from proofs. This explains that some incorrect rules have been able to slip into the works of Āryabhaṭa and Bhāskara and remain undiscovered for a long time.

Before we jump to the conclusion that these shared features are due to the influence of Pāṇinian grammar on classical Indian geometry, some further facts and arguments have to be taken into consideration. The claimed absence of proofs in classical Indian geometry, in particular, has to be confronted with the conflicting claim that proofs existed already in Indian geometry long before Āryabhaṭa and Bhāskara, and presumably before Pāṇini,[69] viz., in the geometry of the Vedic Śulba Sūtras.[70] Frits Staal – whose views about the importance of grammar are central to this essay – maintains this position in a very recent article (1999),[71] but does not provide arguments to bolster it beyond referring to a number of publications[72] by Seidenberg (1978, 1983; one might add 1962, 1975: 289 ff.) and Van der Waerden (1983: 26 ff.; one might add 1980).[73] Seidenberg and Van der Waerden – like Staal himself – argue for a common origin of mathematics, or of geometry specifically, as found in various cultures.[74] Staal, for example, pleads for a common origin of Greek and Vedic geometry in Bactria/Margiana; Van der Waerden proposes "a Neolithic Geometry and Algebra, invented somewhere in Central Europe between (say) 3500 and 2500 B.C., in which the 'Theorem of Pythagoras' played a central rôle" (1980: 29; cp. 1983: 33–35); Seidenberg claims to prove that "[a] common source for the Pythagorean and Vedic mathematics is to be sought either in the Vedic mathematics or in an older mathematics very much like it" (1978: 329; similarly already Schroeder, 1884).[75] More important for our present purposes is that these three authors – as pointed out above – agree that Vedic mathematics had proofs. Van der Waerden goes further and maintains that already the original mathematicians (whom he calls the pre-Babylonian mathematicians) had them (1980: 8): "I suppose that these mathematicians had proofs, or at least plausible derivations. A pupil who has to solve a mathematical problem can do it just by applying a rule he has learnt, but the man who invented the rule must have had some sort of derivation. I also suppose that our pre-Babylonian mathematicians had a proof of the 'Theorem of Pythagoras'." Seidenberg is hardly less brazen (1978: 332): "The striking thing [in the Āpastamba Śulvasūtra] is that we have a proof. One will look in vain for such things in Old-Babylonia. The Old-Babylonians, or their predecessors, must have had proofs of their formulae, but one does not find them in Old-Babylonia."

The notion of proof that is claimed to have existed in India and elsewhere by the above-mentioned authors has been examined by G.E.R. Lloyd in the 3rd chapter of his book *Demystifying Mentalities* (1990), some passages of which are worth quoting. Lloyd begins as follows (p. 74):

> At the outset we must be clear that 'proof' and 'proving' may signify a variety of more or less formal, more or less rigorous, procedures. In some domains, such as law, proving a fact or a point of law will be a matter of what convinces an audience as being beyond reasonable doubt. Again in some contexts, including in mathematics, 'proving' a result or a procedure will sometimes consist simply in testing and checking that it is correct. Both of these are quite informal operations. But to give a formal proof of a theorem or proposition requires at the very least that the procedure used be exact and of general validity, establishing by way of a general, deductive justification the truth of the theorem or proposition concerned. More strictly still Aristotle was to express the view that demonstration in the fullest sense depended not just on deductive (he thought specifically syllogistic) argument but also on clearly identified premises that themselves had to fulfil rather stringent conditions ... He was the first not just in Greece, but so far as we know anywhere, explicitly to define strict demonstration in that way.
>
> Two crucial distinctions have, then, to be observed, (1) between formal proofs and informal ones, and (2) between the practice of proof (of whatever kind) and having an explicit concept corresponding to that practice, a concept that incorporates the conditions that need to be met for a proof to have been given.

Lloyd points out that the second distinction, in his view, has been ignored or badly underplayed in recent attempts to see the notion of proof as originating long before even the earliest extant Egyptian and Babylonian mathematics. He then turns to Vedic mathematics and argues that the authors of its key texts were not concerned with proving their results at all, but merely with the concrete problems of altar construction. Vedic mathematics is again dealt with in a "Supplementary note: geometry and 'proof' in Vedic ritual" (pp. 98–104), where Lloyd observes that "the notion that the authors in question had a clear and explicit *concept* of proof is subject to the general doubt ... that to obtain results is one thing, to have that concept as an explicit one is another. ... It also falls foul of one further fundamental difficulty. This is that no clear distinction is drawn in these texts between the rules that are expressed to arrive at what we should call *approximations* and those that are employed to yield what *we* should call *exact* results" (p. 101).

The question could be asked whether the notion of proof is really culture-specific to ancient Greece. Here Lloyd comments (p. 75):

> The practice of proof, in Greece, antedates by several generations the first explicit formal definition (first given by Aristotle in the fourth century B.C.) and the process whereby such notions as that of the starting-points or axioms came to be clarified was both hesitant and gradual. That long and complex development, in Greece, belongs to and is a further instance of the gradual heightening of self-consciousness we have

exemplified before, when second-order questions came to be raised concerning the nature, status, methods and foundations of different types of inquiry. None of the attendant circumstances surrounding these developments, and none of the steps by which the various interrelated key notions came to be made explicit, can be paralleled in Vedic literature or in the evidence for Vedic society.[76]

Does this mean that we should not expect any such notion to have existed in classical India? Is the notion of proof really a Greek invention, determined by the specific social and political situation prevailing in that culture? And does it follow that all cultures that do possess the notion of proof must have borrowed it – directly or indirectly – from ancient Greece?

Before trying to answer these questions it seems appropriate to recall that the absence of explicit proofs and of an identifiable notion of proof seems to be a common feature of many cultures that produced geometry. O. Neugebauer, for example, states about ancient Babylonia (1957: 45–46; cited in Seidenberg, 1975: 286): "It must ... be underlined that we have not the faintest idea about anything amounting to a 'proof' concerning the relations between geometrical magnitudes."[77] Richard J. Gillings – in an appendix meant to counter some of the negative criticism addressed at the mathematics of the ancient Egyptians on account of its lack of formal proof – concludes nonetheless (1972: 234): "We have to accept the circumstance that the Egyptians did not think and reason as the Greeks did. If they found some exact method (however they may have discovered it), they did not ask themselves *why* it worked. They did not seek to establish its universal truth by an a priori symbolic argument that would show clearly and logically their thought processes."[78] Chinese geometry, it appears, did not use proofs either. When early in the seventeenth century Euclid's Elements were translated into Chinese,[79] by Matteo Ricci and Xu Guangqi, the latter of these two wrote (in the preface to another work) that only after the translation of Euclid's Elements into Chinese had it become possible to transmit proofs and explanations. In fact, he maintained, the Western methods of conveying are not essentially different from the methods transmitted in earlier Chinese treatises. What makes Western mathematics more valuable is that it supplies explanations which show why the methods are correct.[80] Joseph Needham (1959: 91) confirms this by stating: "In China there never developed a theoretical geometry independent of quantitative magnitude and relying for its proofs purely on axioms and postulates accepted as the basis of discussion." We learn from Jean-Claude Martzloff's *A History of Chinese Mathematics* that in the Chinese tradition of geometry "the figures essentially refer not to idealities but to material objects which, when manipulated in an appropriate manner, effectively or in imagination, may be used to make certain mathematical properties visible".[81] This has various consequences, among them the use of empirical methods: "To show that the side of a regular hexagon inscribed in a circle has the same length as the radius, six small equilateral triangles are assembled and the result is determined *de visu*. One proof technique for determination of the volume of the sphere involves weighing it."[82] Sometimes "the reader is asked to put together jigsaw pieces, to look at a figure or to undertake calculations which themselves constitute the sole justification of the matter in hand". Martzloff concludes (p. 72): "If one has to speak of 'proofs', it might be said that, from this point of view, the whole of the mathematician's art consists of making visible those mathematical phenomena which are apparent not in Platonic essences but in tangible things".[83] The concern with mathematical objects as parts of objective reality reminds us of the similar concern on the part of Bhāskara, studied above.

Should we then conclude that the notion of proof only belongs to ancient Greece and its inheritors? I think the situation is more complex than this. Recall that India at the time of Āryabhaṭa and Bhāskara *did* have a clear notion of proof. Such a notion was present in Indian logic, which had been developing since long before Āryabhaṭa and reached some of its peaks in the persons of Dignāga and Dharmakīrti precisely during the period that separates Āryabhaṭa from his commentator. Bhāskara at any rate, it would appear, had no excuse for not being aware of the notion of proof, and for not providing proofs for his theorems. At first sight the situation of these two mathematicians should not therefore be very different from Euclid in Greece. Greek philosophers developed the notion of proof in their logic, and Greek mathematicians did the same in geometry. Indian philosophers developed the notion of proof in their logic, but the Indian mathematicians did not follow suit. Why not? The situation becomes even more enigmatic if we assume that Indian geometry derives from Greek geometry (see note 8, above), and that therefore some Indian mathematicians (presumably well before Āryabhaṭa and Bhāskara) had been familiar with Euclidean procedures.

Here it is important to recall that Indian philosophers of the classical period were engaged in an ongoing debate with each other, in which radically different positions were defended and criticized. This debate went on because all participants were part of what I have called a "tradition of rational inquiry",[84] which translated itself in the social obligation – partly embodied in kings and their courts – to listen to one's critics and defend one's own point of view. This is not the place to discuss the enormous impact which this tradition of rational inquiry has had on the shape and direction of Indian philosophy, but it is clear that the development of logic in the different schools of philosophy was a result of this ongoing confrontation. Logic specified the rules which even one's greatest enemy would have to accept.

It seems likely that those who studied and practiced mathematics in India were not to the same extent as their philosopher contemporaries obliged to defend their positions against opponents who disagreed with practically every single word they uttered (or wrote down).[85] There were differences of opinion, to be sure,[86] but they were apparently not looked upon as particularly threatening. Bhāskara could show that the Jaina value for π, or rather their formula for calculating the length of arc that used this value, could not be correct. But apparently such disagreements were not yet considered sufficiently serious to rethink the whole system. Indeed we have seen that Bhāskara does not mind citing an apparently Jaina rule to justify a calculation. Indian astronomers and mathematicians, it appears, were not engaged in any such ongoing debate with fierce opponents belonging to altogether different traditions as were the philosophers.[87] Nor were they – it seems – particularly interested in what was going on in philosophy. Randall Collins claims (1998: 551) that there are no recorded contacts between philosophical and mathematical networks, and that no individuals overlap both activities. This may not be entirely correct: David Pingree (private communication) mentions in this connection Nīlakaṇṭha Somayājin's *Jyotirmīmāṃsā* (written in 1504; cp. Pingree, 1981: 50, 128); and Jean Michel Delire – in a paper read at the XIth World Sanskrit Conference, Turin, April 2000 – has drawn attention to Veṅkaṭeśvara Dīkṣita, a late 16th century commentator who combined skills in Mīmāṃsā and mathematics (Pingree, 1981: 6, 129).[88] Yet Collins's claim appears to hold true for Bhāskara (and perhaps other mathematicians of his time), judging by the list of authorities cited by him which is given at the end of the edition of his Āryabhaṭīya Bhāṣya (Shukla, 1976: 345–346). Bhāskara often cites grammatical and generally linguistic texts (as we have seen), astronomical texts, some religious and literary treatises, but not a single philosophical work.[89] It is true that we have very little information about the lives and circumstances of the early Indian mathematicians, but there is no reason that I know of to doubt their relative intellectual isolation,[90] combined perhaps with low social esteem.[91] It goes without saying that further research in this complex of questions is called for.

Some scholars are of the opinion that more recent Indian mathematical authors **did** provide proofs for geometrical theorems. Takao Hayashi, for example, remarks (1995: 75): "the term *upapatti* stands for the proof or derivation of a mathematical formula. We find a number of instances of *upapatti* used in that sense in later commentaries such as Gaṇeśa's *Buddhivilāsinī* (A.D. 1545) on the *Līlāvatī* and Kṛṣṇa's *Navāṅkura* (ca. A.D. 1600) on the *Bījagaṇita*." M.D. Srinivas (1990) gives a list of commentaries that contain mathematical *upapattis* in an appendix (no. I; p. 57 ff.) to his article; all of them date from the 16th and 17th centuries. The highly interesting question to what extent these later mathematical authors had an explicit concept of proof (not necessarily Euclidean; cp. Lloyd's remarks, cited above) and, if so, when, how and why such a concept made its appearance in Indian mathematical works cannot be addressed in this article. One may wonder whether the type of arguing that had become common in philosophical debate slowly found its way into this area of knowledge. But whether or not such a shift of attitude took place in mathematics, there seems to be no doubt that Bhāskara I, the commentator whose work we are considering, was not (yet) affected by it.

The following example confirms this. The Āryabhaṭīya Bhāṣya – at least in the interpretation of Agathe Keller, who attributes her interpretation to a suggestion made by Takao Hayashi – contained the following diagram "to convince the dull-witted":[92]

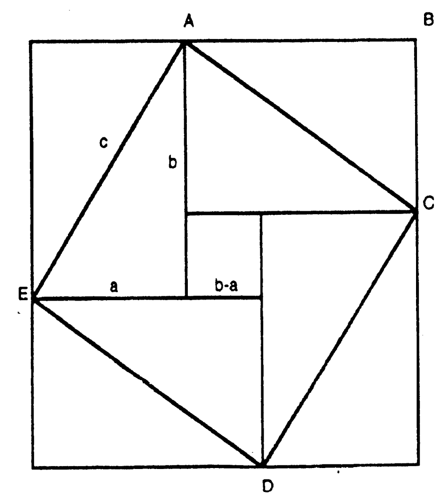

Figure 3. There is a square corresponding to AC^2.

Bhāskara uses this diagram to show that the square of a diagonal of a rectangular surface does indeed correspond to a geometric square: AC^2 (= AB^2 + BC^2) corresponds to the surface ACDE. However, this same diagram could easily be used, and has been used by more recent authors, to prove the Pythagorean theorem (though not, of course, in a "Euclidean" manner). In this case, as Keller points out in her thesis, the area of the interior small square (whose sides are equal to $b - a$) increased by the area of the four triangles (whose sides are a and b), gives the area of the big square whose sides are the hypotenuse of the four triangles (in other words: $c^2 = (b-a)^2 + 4 \times ab/2 = a^2 + b^2$. This proof is given by the author Gaṇeśa Daivajña in around 1545 (Srinivas, 1990: 39) and, it is claimed, by Bhāskara II around 1150.[93] Bhāskara I had this proof of the Pythagorean theorem so to say under his nose, but he was apparently not interested.

Where does all this leave us with regard to the influence of Pāṇini's grammar on geometry in India? Those who, like Staal and others, are inclined to look upon Pāṇini as having provided methodical guidance to the Indian sciences, will find features in Āryabhaṭa and Bhāskara that remind them of Pāṇini's grammar and which they might like to attribute to the latter's influence on the former. If so, they may have to consider whether and to what extent this influence is responsible for the absence of proofs in early classical geometry. A comparative study of various ancient cultures shows that proofs in geometry do not normally appear, unless social, political or other circumstances have led to the kind of awareness in which an explicit notion of proof has found its place. Such a notion of proof did exist in classical India at the time of Āryabhaṭa and Bhāskara, but among philosopher-logicians, not among mathematicians. This peculiar situation may in part have to be explained by the fact that mathematicians were less exposed to debate and controversy than the philosophers. To this must be added that the Indian mathematicians may have been happy with their methods which – if not in the case of Āryabhaṭa and Bhāskara, but certainly in that of Brahmagupta and others – led to remarkable results.[94] This does not change the fact that the example of the grammar of Pāṇini may conceivably have lent added respectability to a geometry without proof, even at a time when mathematicians came across this notion (whether under Greek mathematical or Indian philosophical influence). Mathematics in the style of Āryabhaṭa and Bhāskara had to be good enough, for it resembled in some essential respects Pāṇini's grammar, which certainly was good enough.

We are therefore led to the following conclusion. There is no proof for the claimed methodical guidance of Pāṇini's grammar with respect to classical Indian geometry, except perhaps where completely external features of presentation are concerned. Contentwise, classical Indian geometry contrasts as sharply with Euclidean geometry as do other pre-modern geometries – in China, Babylonia and Egypt – which had no knowledge of Pāṇini's grammar. Like these other pre-modern geometries, classical Indian geometry did not use proofs. The most noteworthy distinguishing feature of classical Indian geometry is that, unlike the other pre-modern geometries, it developed in surroundings where the notion of proof was well-established. It is not immediately clear whether this lack of susceptibility to the notion of proof on the part of classical Indian geometry is in need of an explanation. If it is, Pāṇini's grammar might conceivably be enumerated among the factors that played a role. In that case one may have to consider the possibility that the influence of Pāṇini's grammar, far from encouraging the development of an abstract geometry, had the opposite effect.

NOTES

* This article has been inspired by the recent thesis of Agathe Keller (2000). I have further profited from comments on an earlier version by Pascale Hugon, David Pingree and Kim Plofker. The responsibility for the opinions expressed and for mistakes remains mine.

[1] Cp. further note 9, below. Filliozat (1995: 40) observes: "Obviously in the mind of the Indian learned men the study of language held the place which mathematics held in the mind of the ancient Greek philosophers." See also Ruegg, 1978.

[2] Cp. Pingree, 1981: 2: "classical astronomy and mathematics had virtually ceased to be studied or taught by the end of the nineteenth century. A new group of Indian and foreign scholars has, however, begun to work in these areas since World War II".

[3] The reference is to B.V. Subbarayappa and K.V. Sarma, *Indian Astronomy: A Source Book*, Bombay 1985.

[4] Note further that "[a]ccording to Pingree's estimation ... there still exist today some 100,000 Sanskrit manuscripts in the single field of *jyotiḥśāstra* (astronomy, astrology, and mathematics)" (Yano, 1987: 50).

[5] On Indian medical literature see, in particular, Meulenbeld, 1999 ff.

[6] I am aware of two exceptions. One is Singh, 1990, which tries to show, not very convincingly, "that algebraic inclination of ancient Indian mathematics was strongly related to foundational attitude developed by linguistic tradition" (p. 246). The other exception is Filliozat, 1995.

[7] For some reflections on the limited influence exerted by Pāṇini's grammar on philosophy in India, see Bronkhorst, forthcoming.

[8] On the Greek influence on Indian astronomy, see Pingree, 1971; 1993. On the extent to which Greek geometry influenced Indian geometry, see also Shukla's remark cited in note 47, below.

[9] Cp. Joseph, 1991: 217–219: "An indirect consequence of Pāṇini's efforts to increase the linguistic facility of Sanskrit soon became apparent in the character of scientific

and mathematical literature. This may be brought out by comparing the grammar of Sanskrit with the geometry of Euclid – a particularly apposite comparison since, whereas mathematics grew out of philosophy in ancient Greece, it was, as we shall see, partly an outcome of linguistic developments in India. The geometry of Euclid's Elements starts with a few definitions, axioms and postulates and then proceeds to build up an imposing structure of closely interlinked theorems, each of which is in itself logically coherent and complete. In a similar fashion, Panini began his study of Sanskrit by taking about 1700 basic building blocks – some general concepts, vowels and consonants, nouns, pronouns and verbs, and so on – and proceeded to group them into various classes. With these roots and some appropriate suffixes and prefixes, he constructed compound words by a process not dissimilar to the way in which one specifies a function in modern mathematics. Consequently, the linguistic facility of the language came to be reflected in the character of mathematical literature and reasoning in India."

[10] Mahā-bh I p. 6 ll. 3–7: *katham tarhīme śabdāḥ pratipattavyāḥ/ kiṃcit sāmānyaviśeṣavat lakṣaṇam pravartyaṃ yenālpena yatnena mahato mahataḥ śabdaughān pratipadyeran/ kiṃ punas tat/ utsargāpavādau/ kaścid utsargaḥ kartavyaḥ kaścid apavādaḥ/ kathaṃjātīyakaḥ punar utsargaḥ kartavyaḥ kathaṃjātīyako 'pavādaḥ/ sāmānyenotsargaḥ kartavyaḥ/.*

[11] Cp. Prakash Sarasvati, 1986: 157: "for the first time we find Āryabhaṭa ... in his Āryabhaṭīya describing [mathematics] as a special section (Gaṇitapāda). Brahmagupta ... also followed Āryabhaṭa in this respect and gave the science of calculation (*gaṇita*) a special place in his treatise on astronomy. The Siddhānta treatises earlier than those of Āryabhaṭa and Brahmagupta do not contain a chapter exclusively devoted to *gaṇita* (the Sūrya-Siddhānta and the Siddhāntas of Vasiṣṭha, Pitāmaha and Romaka are thus without *gaṇita* chapters)." Also Hayashi, 1995: 148: "Bhāskara (A.D. 629) tells us that Maskarī, Pūraṇa, Mudgala and other '*ācāryas*' (teachers) composed mathematical treatises, but none of them is extant now. We can only have a glimpse of Indian mathematics of those early times through the extant astronomical works of Āryabhaṭa, Bhāskara, and Brahmagupta, only a few chapters of which are devoted to mathematics proper." It should of course not be forgotten that much geometry is to be found in texts dealing with mathematical astronomy.

[12] Āryabhaṭīya Bhāṣya p. 44 l. 18: *gaṇitaṃ dviprakāram: rāśigaṇitaṃ kṣetragaṇitam.* Cp. Hayashi, 1995: 62.

[13] Āryabhaṭīya 2.17ab: *yaś caiva bhujāvargaḥ koṭīvargaś ca karṇavargaḥ saḥ.*

[14] Filliozat, 1988: 255–256. Another way of looking at the same characteristic is expressed in Pingree, 1978: 533, which speaks about the corrupt tradition of the earliest surviving Sanskrit texts: "The cause of this corruption is usually that the texts had become unintelligible; and this unintelligibility is not unrelated to the style developed by Indian astronomers. The texts proper were composed in verse in order to facilitate memorization, with various conventions for rendering numbers into metrical syllables. The exigencies of the meter often necessitated the omission of important parts of mathematical formulas, or contributed to the imprecision of the technical terminology by forcing the poet to substitute one term for another. ..."

[15] Filliozat, 1995: 46 mentions the "high degree of generality" – which includes a general formulation of the Pythagorean theorem, "the first theorem enunciated in the history of Sanskrit mathematics" (p. 48) – as "a quality which [a Śulba-sūtra] shares only with Pāṇini's model".

[16] Sharma, 1966, vol. III p. 829 (= Brāhmasphuṭasiddhānta 12.24): *karṇakṛteḥ koṭikṛtiṃ viśodhya mūlaṃ bhujo bhujasya kṛtim/ prohya padaṃ koṭiḥ koṭibāhukṛtiyutipadaṃ karṇaḥ//.* Tr. Colebrooke, 1817: 298. As a matter of fact, the Pythagorean theorem occurs several times, in several guises, in this portion of the text; e.g. 12.22cd: *svābādhāvargonād bhujavargān mūlam avalambaḥ* (tr. Colebrooke: "The square-root, extracted from the difference of the square of the side and square of its corresponding segment of the base, is the perpendicular"); 12.23cd: *karṇakṛtir bhūmukhayutidalavargonā padaṃ lambaḥ* (tr. Colebrooke: "[In any tetragon but a trapezium,] subtracting from the square of the diagonal the square of half the sum of the base and summit, the square-root of the remainder is the perpendicular"); 12.42ab: *jyāvyāsakṛtiviśeṣān mūlavyāsāntarārdham iṣur alpaḥ* (tr. Colebrooke: "Half the difference of the diameter and the root extracted from the difference of the squares of the diameter and the chord is the smaller arrow").

[17] Bronkhorst, forthcoming 2: § 2.

[18] An exception *may* have to be made for the examples accompanying the sūtras of the Bakhshālī Manuscript.

[19] This is the name here adopted for the combined text consisting of the Daśagītikāsūtra-vyākhyā and the Āryabhaṭa-tantra-bhāṣya, expressions apparently used by Bhāskara himself to designate the two parts of his commentary (on chapter 1 and on the remaining chapters respectively); see Shukla, 1976: xlix.

[20] The Gaṇitapāda of Bhāskara's Āryabhaṭīya Bhāṣya has recently been studied and translated by Agathe Keller (2000). Reading her thesis has inspired me to write this article and helped me in the interpretation of many passages.

[21] It is known that a number of classical commentaries imitated the style of the Mahābhāṣya, sometimes calling themselves "Vārttika" (Bronkhorst, 1990). Bhāskara's Bhāṣya does not adopt this so-called "Vārttika-style".

[22] Āryabhaṭīya Bhāṣya p. 96 l. 15: *yaś ca bhujāvargaḥ yaś ca koṭivargaḥ etau vargau ekatra karṇavargo bhavati.*

[23] In order to avoid misunderstanding it must here be emphasized that I am aware of the fact that proof (Euclidean or other) does not appear to be essential to geometry in many cultures; see below. The present discussion on the absence of proof in Bhāskara and elsewhere finds its justification in the comparative approach adopted in this article, inspired by the remarks of Ingalls and Staal cited in its first pages.

[24] Āryabhaṭīya Bhāṣya p. 44 l. 17: *karṇabhujayoḥ samatvaṃ karoti yasmāt tataḥ karaṇī*; tr. Hayashi, 1995: 62. The dual is strange, and one wonders whether the original reading may not have been: *bhujayoḥ karṇasamatvaṃ karoti yasmāt tataḥ karaṇī.* For a discussion of the term *karaṇī* in various mathematical texts, see Hayashi, 1995: 60–64.

[25] Gaṇitapāda 17cd: *vṛtte śarasaṃvargo 'rdhajyāvargaḥ sa khalu dhanuṣoḥ.*

[26] Āryabhaṭīya Bhāṣya p. 103 ll. 12–13: *pratyayakaraṇam ca sarveṣv eva kṣetreṣu "yaś caiva bhujāvargaḥ koṭīvargaś ca karṇavargaḥ saḥ" ity anenaiveti.*

[27] Earlier authorities have already drawn attention to the absence of proofs in early classical Indian mathematics. So e.g. Kline, 1972: 190, as cited in Srinivas, 1990: 76 n. 2: "There is much good procedure and technical facility, but no evidence that [the Hindus] considered proof at all." See however below.

[28] Parallel to the hypothesis here considered to the extent that Pāṇini may have exerted a negative influence on the development of certain sciences in India, is Lloyd's (1990: 87 ff.) observation that the influence of Euclid's Elements on the development of Greek science was not only positive. Lloyd draws attention in particular to medicine and physiology, certain areas of mathematics itself, and to the extent to which problems in physics and elsewhere had a tendency to be idealized (exactness may be obtained only at the cost of applicability).

[29] This enumeration is not exhaustive even with regard to passages identified in the text. Passages from the Mahābhāṣya are identified p. 3 ll. 7–9, p. 8 ll. 1–2, p. 23 ll. 25–26; but they are not mentioned in the appendix.

[30] It is tempting to see, furthermore, in the line *loke ca na so 'sti gaṇitaprakāraḥ yo 'yaṃ vṛddhyātmako 'pacayātmako vā na bhavati* (p. 44 ll. 6–7; introducing Gaṇitapāda 1) a reflection of Vkp 1.131ab: *na so 'sti pratyayo loke yaḥ śabdānugamād ṛte.*

Cp. however *na so 'sti puruṣo loke yo na kāmayate śriyam* in Jagajjyotirmalla's Ślokasārasaṃgraha, quoted from the Hitopadeśa (Lindtner, 2000: 60, 65).

[31] Many of the following quotations from the Mahābhāṣya could be identified thanks to the electronic version of that text prepared by George Cardona. No attempt has been made to identify all quotations from the Mahābhāṣya.

[32] Mahā-bh II p. 246 l. 6: *sāmānyacodanās tu viśeṣeṣv avatiṣṭhante.*

[33] Mahā-bh I p. 5 l. 28: *yaḥ sarvathā ciraṃ jīvati sa varṣaśataṃ jīvati.*

[34] Mahā-bh III p. 32 ll. 4–5.

[35] Mahā-bh II p. 437 l. 2. Mahā-bh II p. 218 ll. 15–16 (on Pāṇ. 4.1.48 vt. 3) gives as example of *tātsthyāt*: *mañcā hasanti*, where Bhāskara gives *mañcāḥ krośanti.*

[36] The Kāśikā on Pāṇ. 6.4.11 has *samāsānto vidhir anityaḥ* with variant reading *samāsāntavidhir anityaḥ.*

[37] Compare Āryabhaṭīya Bhāṣya p. 7 ll. 7–8 (*etad ekaikasya granthalakṣaṇalakṣyaṃ maskaripūraṇamudgalaprabhṛtibhir ācāryair nibaddhaṃ kṛtaṃ* ...) with Mahā-bh I p. 12 ll. 15–17 (*lakṣyalakṣaṇe vyākaraṇam* [vt. 14]/ *lakṣyaṃ ca lakṣaṇaṃ caitat samuditaṃ vyākaraṇaṃ bhavati/ kiṃ punar lakṣyaṃ lakṣaṇaṃ ca/ śabdo lakṣyaḥ sūtraṃ lakṣaṇam/*).

[38] David Pingree (private communication) informs me that this employment of *sūtra* became common in commentaries on mathematical and astronomical texts. In other disciplines the word *sūtra* refers much less commonly to a metrical mūla text; two texts that do so are the Yuktidīpikā (on the Sāṃkhyakārikā) and the Abhidharmakośabhāṣya.

[39] They can coincide with verses, in which case Bhāskara, in the first chapter, uses the expression *gītikāsūtra* (or *gītikasūtra, gītisūtra*; e.g. Āryabhaṭīya Bhāṣya p. 1 ll. 10–11, p. 7 ll. 13 & 16, p. 11 ll. 14 & 20, etc.), elsewhere *āryāsūtra* (e.g. p. 247 l. 20).

[40] Āryabhaṭīya Bhāṣya p. 105 ll. 12–17; p. 107 ll. 10–11.

[41] So Hayashi, 1994: 122: "Neither the Āryabhaṭīya nor the Brāhmasphuṭasiddhānta contains proofs of their mathematical rules, but this does not necessarily mean that their authors did not prove them. It was probably a matter of the style of exposition."

[42] D.E. Smith (1923: 158) claims to find faulty theorems in Brahmagupta's Brāhmasphuṭasiddhānta, but most of his cases concern a rule (12.21ab) which is presented as approximate (*sthūla*) by its author. His one remaining case – a formula for the area of quadrilaterals that is presented as being valid without restriction, but is in reality only valid for cyclic quadrilaterals (12.21cd) – may have to be interpreted differently: J. Pottage, at the end of a detailed study (1974), reaches the following conclusion (p. 354): "I have been unable to accept that Brahmagupta could have imagined that his rules would apply to all quadrilaterals whatsoever".

[43] Āryabhaṭīya Gaṇitapāda 6cd: *ūrdhvabhujātatsaṃvargārdhaṃ sa ghanaḥ ṣaḍaśrir iti.*

[44] Cp. Āryabhaṭīya Gaṇitapāda 7: *samapariṇāhasyārdhaṃ viṣkambhārdhahatam eva vṛttaphalam/ tannijamūlena hataṃ ghanagolaphalaṃ niravaśeṣam//* "Half the even circumference multiplied by half the diameter is precisely the fruit (i.e., the area) of a circle. That (the area) multiplied by its own square root is the exact volume (lit. the without-a-remainder solid fruit) of a sphere." (tr. Hayashi, 1997: 198; similarly Clark, 1930: 27). Bhāskara also provides an approximate, "practical" (*vyāvahārika*), rule for calculating the volume of a sphere (p. 61 l. 27): *vyāsārdhaghanaṃ bhittvā navaguṇitam ayoguḍasya ghanagaṇitam* "Having divided [into two] the cube of half the diameter multiplied by nine, the calculation of the volume of an iron ball [has been carried out]."

[45] Filliozat and Mazars (1985) mention Conrad Müller (1940) and Kurt Elfering (1968, 1975, 1977).

[46] See also Hayashi, 1997: 197–198. Smeur, 1970: 259–260 presents a hypothesis to explain how the incorrect rule for the volume of a sphere might have come into being.

[47] Keller (2000: 188) suggests the following explanation: "Bhāskara semble considérer qu'il y a une continuité entre la figure plane et la figure solide. Cette continuité pourrait servir d'explication aux formules fausses de l'Āryabhaṭīya. Ainsi, la lecture du calcul sur le volume du cube repose sur la lecture du vers qui fournit l'aire du carré. Le volume du cube est le produit de l'aire du carré par la hauteur ($V = A \times H$). De même le volume de la sphère est la racine carrée de l'aire multipliée par l'aire ($V = A \times \sqrt{A}$). Le volume de la Sphère semble encore une fois être le produit d'une aire et d'une «hauteur» que représente numériquement la racine-carré de l'aire. L'aire d'un triangle équilatéral est le produit de la moitié de la base et d'une hauteur ($A = 1/2\,B \times H$). Dans la continuité de cette aire, le volume du tétraèdre est donné avec la même pondération: la moitié de l'aire du triangle équilatéral et de la hauteur ($V = 1/2\,A \times H$)." Similarly Plofker, 1996: 62: "This error ... suggests that in this case reasoning by analogy led Āryabhaṭa astray." Shukla (1972: 44) observes: "It is strange that the accurate formula for the volume of a sphere was not known in India. This seems to suggest that Greek Geometry was not known at all in India ..."

[48] Contrast the errors of Āryabhaṭa and Bhāskara with the situation in classical Greece: "One of the most impressive features of Greek mathematics is its being practically mistake-free. An inspectable product in a society keen on criticism would tend to be well tested." (Netz, 1999: 216)

[49] Āryabhaṭīya Bhāṣya p. 11 l. 23 – p. 12 l. 1: *anenācāryeṇa mahadbhis tapobhir brahmārādhitaḥ/ ... / ato 'nena lokānugrahāya sphuṭagrahagatyarthavācakāni daśa gītikāsūtrāṇi gaṇitakālakriyāgolārthavācakam āryāṣṭaśataṃ ca vinibaddham/.*

[50] The Greek situation tends to be idealized. A closer study of the evidence leads Netz (1999: 95) to the following assessment: "Greek mathematical works do not start with definitions. They start with second-order statements, in which the goals and the means of the work are settled. Often, this includes material we identify as 'definitions'. In counting definitions, snatches of text must be taken out of context, and the decision concerning where they start is somewhat arbitrary."

[51] Āryabhaṭīya Bhāṣya p. 77 ll. 9–15: *vayaṃ tu brūmaḥ: asti kāṣṭhatulyajyeti/ yadi kāṣṭhatulyajyā na syāt tadā samāyām avanau vyavasthānam evāyoguḍasya na syāt/ tenānumīmahe kaścit pradeśaḥ so 'stīti yenāsāv ayoguḍaḥ samāyām avanāv avatiṣṭhate/ sa ca pradeśaḥ paridheḥ ṣaṇṇavatyaṃśaḥ/ kāṣṭhatulyajyā 'nyair apy ācāryair abhyavagatā: tatparidheḥ śatabhāgaṃ spṛśati dharāṃ golakaśarīrāt iti.*

[52] The quoted line is problematic, not only because of the neuter ending of *śatabhāgaṃ*, but even more so on account of *golakaśarīrāt* where one would expect something like *golakaśarīratvāt*. Shukla (1976: lxiv) translates: "Due to the sphericity of its body, a sphere touches the Earth by one-hundredth part of its circumference"; Hayashi (1997: 213) has: "A hundredth part of its circumference touches the ground because of its having a spherical body."

[53] Kim Plofker reminds me that one cannot balance a circle on level ground because it would fall over sideways; no confusion may therefore be involved here.

[54] Mahā-bh I p. 7 l. 28 – p. 8 l. 1: *ghaṭena kāryaṃ kariṣyan kumbhakārakulaṃ gatvāha kuru ghaṭaṃ kāryam anena kariṣyāmīti/ na tadvac chabdān prayokṣyamāṇo vaiyākaraṇakulaṃ gatvāha kuru śabdān prayokṣya iti/.*

[55] Mahā-bh I p. 11 ll. 1–2.

[56] Concerning the nature of the objects of Greek geometry, read the following remarks by Reviel Netz (1999: 54–56): "On the one hand, the Greeks speak as if the object of the proposition is the diagram. ... On the other hand, Greeks act in a way which precludes this possibility ... Take Pünktchen for instance. Her dog is lying in her bed, and she stands next to it, addressing it: 'But grandmother, why have you got such large teeth?' What is the semiotic role of 'grandmother'?

It is not metaphorical – Pünktchen is not trying to insinuate anything about the grandmother-like (or wolf-like) characteristics of her dog. But neither is it literal, and Pünktchen knows this. Make-believe is a *tertium* between literality and metaphor: it is literality, but an as-if kind of literality. My theory is that the Greek diagram is an instantiation of its object in the sense in which Pünktchen's dog is the wolf – that the diagram is a make-believe object; it is functionally identical to it; what is perhaps most important, it is never questioned. ... [The text] does not even hint *what*, ultimately, its objects are; it simply works with an ersatz, as if it were the real thing ... Undoubtedly, many mathematicians would simply assume that geometry is about spatial, physical objects, the *sort* of thing a diagram is. Others could have assumed the existence of mathematicals. The centrality of the diagram, however, and the roundabout way in which it was referred to, meant that the Greek mathematician would not have to speak up for his ontology."

Whatever the Greeks may have thought (or not thought) about the objects of geometry, geometry came to be looked upon as dealing with abstractions, so much so that the expression "Euclid myth" has been coined to designate the false conviction that these objects have anything to do with the outside world: "What is the Euclid myth? It is the belief that the books of Euclid contain truths about the universe which are clear and indubitable. Starting from self-evident truths, and proceeding by rigorous proof, Euclid arrives at knowledge which is certain, objective and eternal. Even now, it seems that most educated people believe in the Euclid myth. Up to the middle or late nineteenth century, the myth was unchallenged. Everyone believed it. It has been the major support for metaphysical philosophy, that is, for philosophy which sought to establish some a priori certainty about the nature of the universe." (Davis and Hersh, 1981: 325, cited in Srinivas, 1990: 81 n. 14). For a recent discussion of the objects of (modern) mathematics, see Shapiro, 1997.

57 Cp. Āryabhaṭīya Bhāṣya p. 48 ll. 15–16: *ardhāyatacaturaśratvāt tribhujasya.*

58 The procedure illustrated in fig. 2 is close to the one which the Chinese mathematician Liu Hui justifies with the reason "Use the excess to fill up the void" (Chemla, 1999: 96 ff.). Chemla (1997) distinguishes between formal *proofs* (in italics) and proofs that are provided "in order to *understand* the statement proved, to know *why* it is true and not only *that* it is true" (p. 229).

59 Āryabhaṭīya Bhāṣya p. 72 ll. 10–17: *athāsannaparidhiḥ kasmād ucyate, na punaḥ sphuṭaparidhir evocyate? evaṃ manyante: sa upāya eva nāsti yena sūkṣmaparidhir ānīyate/ nanu cāyam asti: vikkhaṃbhavaggadasaguṇakaraṇī vaṭṭassa parirao hodi (viṣkambhavargadaśaguṇakaraṇī vṛttasya pariṇāho bhavati) iti/ atrāpi kevala evāgamaḥ naivopapattiḥ/ rūpaviṣkambhasya daśa karaṇyaḥ paridhir iti/ atha manyate pratyakṣeṇaiva pramīyamāṇo rūpaviṣkambhakṣetrasya paridhir daśa karaṇya iti/ naitat, aparibhāṣitapramāṇatvāt karaṇīnām/ ekatrivistārāyāmāyatacaturaśrakṣetrakarṇena daśakaraṇikenaiva tadviṣkambhaparidhir veṣṭyamāṇaḥ sa tatpramāṇo bhavatīti cet tad api sādhyam eva/.* For the use of *karaṇī*, see the reference in note 24 above. Note that Bhāskara's edited text has a number of grammatically incorrect occurrences of °*karaṇika-* and *karaṇitva-*; cp. Pāṇ 7.4.14 *na kapi.*

60 The saying is in Prakrit and has obviously been borrowed from a Jaina text or context.

61 David Pingree (private communication) suggests "that Bhāskara's problem was that no *upapatti* could verify that $\pi = \surd 10$ because of the difficulty of relating the square root of a surd to any previously verified theorem".

62 Hayashi's (1994: 123) following remark is therefore to be read with much caution: "The recognition of the importance of proofs dates back at least to the time of Bhāskara I ... , who, in his commentary on the Āryabhaṭīya, rejected the Jaina value of π, $\surd 10$, saying that it was only a tradition (*āgama*) and that there was no derivation of it."

63 Āryabhaṭīya Bhāṣya p. 59 l. 3: ... *iti trairāśikopapattipradarśanārthaṃ kṣetranyāsaḥ.*

64 Āryabhaṭīya Bhāṣya p. 74 l. 9 – p. 75 l. 4: *pṛṣṭhānayanam api ca daśakaraṇīparidhiprakriyāparikalpanayā sadā na [bhavati/ yataḥ] pṛṣṭhānayane sūtram āryārdham:* ***jyāpādaśarārdhayutiḥ svaguṇā [daśasaṅguṇā karaṇyas tāḥ]*** *[atroddeśakaḥ: dvipañcāśadviṣkambhe dvir avagāhya/] "ogāhūṇaṃ vikkhaṃbham" ity anena jyā labdhā viṃśatiḥ [20]/ [anayā jyayā] pṛṣṭhānayanam: jyāpādaḥ 5, śarārdhaṃ [1], yutiḥ 6, svaguṇā 36, daśasaṅguṇā 360, etā karaṇyaḥ pṛṣṭham/ sakalajyāvargaś catvāri śatāni, pṛṣṭhaṃ karaṇīnāṃ ṣaṣṭiśatatrayam iti, katham etat saṃghaṭate? jyāyasā jyātaḥ pṛṣṭhena bhavitavyam/ tad etad vicāryamāṇam atyantasūkṣmavādināṃ jyātaḥ pṛṣṭham alpīy[o]mānam āpatitam/ ato 'syai avicāritamanoharāyai namo 'stu daśakaraṇyai/.* The edition has *alpīyamānam.*

65 The rule – which reads in full: *ogāhūṇaṃ vikkhambhaṃ egāheṇa saṃguṇaṃ kuryāt/ caüguṇiassa tu mūlaṃ jīvā savvakhattāṇam//* – is similar to Pādalipta's Jyotiṣkaraṇḍaka 191, which has: *ogāhūṇam vikkhaṃbha mo tu ogāhasaṃguṇaṃ kujjā/ catuhi guṇitassa mūlaṃ sā jīvā va'ttha ṇātavvā//.*

66 The Tattvārthādhigama Bhāṣya of Umāsvāti on sūtra 3.11 (I p. 258 ll. 17–18) contains a different rule for the arc: "The arc (a) is the square root of six times the square of the sagitta (s) plus the square of the chord (c) ($a = \surd(6s^2 + c^2)$)" (*iṣuvargasya ṣaḍguṇasya jyāvargayutasya mūlaṃ dhanuḥkāṣṭham*); later authors (Mahāvīra, Āryabhaṭa II) accepted again different rules. See Datta, 1929: 694, 699.

67 For the approximation proposed by Heron of Alexandria, see Heath, 1921: II: 331.

68 He could hardly have such a formula in view of his conviction that "there is a chord equal to the arc [which it subtends]"; see above.

69 Michaels (1978: 56) points out that many terms related to the layering of the altar (the background of Vedic geometry) are known to Pāṇini. In a note he mentions, or refers to, *iṣṭakā, iṣṭakacit, agnicit, āṣāḍhā, aśvinī, vayasā,* etc.

70 Michaels's (1978: 70 ff.) attempts to show that the Śulba Sūtras contain theoretical statements about ideal objects ("theoretische Sätze über ideale Gegenstände") have to be treated with much caution, in the light of what we now know about classical Indian geometry and the geometry in other cultures (see below). About the relationship between Śulba Sūtras and classical Indian geometry, see Kaye, 1919: 3 ("Les oeuvres de la seconde période [= Āryabhaṭa etc.] ne font aucune allusion à un seul de ces sujets des Śulvasūtras") and the qualification added by Michaels (1978: 106: "Allerdings sind insbesonders angesichts dessen, dass einige Termini der [Śulba Sūtras] in der jüngeren indischen Mathematik, wenn auch mit teilweise neuer Bedeutung fortleben, hier gewisse Einschränkungen zu erheben").

71 This does not withhold him from stating (Staal, 1999: 113): "The only Indian counterpart to Euclid is the derivational system of Pāṇini's Sanskrit grammar."

72 To the publications mentioned by Staal one might add Michaels, 1978: 96 ff., and the literature referred to on p. 97 n. 1.

73 Staal does not appear to address the question why in India – which purportedly had had both Pāṇinian grammar and Euclid-like geometry – only the former came to play an important role in science and philosophy. Note in this connection that Āryabhaṭīya Bhāṣya p. 13 l. 24 – p. 16 l. 24 contains a long discussion purporting to show the greater importance of the study of Jyotiṣa than that of grammar.

74 Reflections about the origin(s) of mathematics may have to take into consideration the extent to which mathematical activities were and are present outside the "higher" cultures, in societies without writing. See in this connection Marcia Ascher's book *Ethnomathematics* (1991); further Ascher, 1994.

75 Cp. also Friberg, 1990: 580: "There are reasons to believe that Babylonian mathem-

atics in a decisive way influenced Egyptian, Greek, Indian and Chinese mathematics, in form as well as in content, during the last half of the – 1st mill., if not earlier."

76 Lloyd sums up his views on the development of proof in the following passages (pp. 95–96): "We can say that the development, in Greece, of the demand for *certainty* sprang in part from a dissatisfaction shared by a variety of individuals with the merely persuasive. ... We have related other intellectual developments that took place in early Greek thought ... to the political and social background, for example the extensive experience that many Greeks had of evaluating arguments in the law courts and assemblies. ... [W]e should conclude that [in the development of formal or rigorous proof] too ... the political and legal background plays a role at least at the beginning of what might otherwise seem a merely intellectual development. However, the qualification to the thesis that must be entered is that, in this instance, that role was not as a source of positive, but rather of negative, models."

77 Cp. Friberg, 1990: 583: "it is clear that the Greek mathematicians completely transformed the intellectual goods they received [from Babylonian mathematics]. ... Rigorous *proofs* based on *abstract* definitions and axioms took over the role played in Babylonian mathematics by a conceptually simpler method, that of using a *reversal of the steps* in an algorithm with given numerical data in order to *check* the computed values."

78 G.G. Joseph – in a chapter called "Egyptian and Babylonian mathematics: an assessment" – proposes to adjust the notion of proof so as to include these traditions (1991: 127): "A modern proof is a procedure, based on axiomatic deduction, which follows a chain of reasoning from the initial assumptions to the final conclusion. But is this not taking a highly restrictive view of what is proof? Could we not expand our definition to include, as suggested by Imre Lakatos ... , explanations, justifications and elaborations of a conjecture constantly subjected to counter-examples? Is it not possible for an argument or proof to be expressed in rhetoric rather than symbolic terms, and still be quite rigorous?" If one is determined to find proofs in all cultures that had geometry or something resembling it, adjusting the notion of proof may be the way to succeed; it is however open to doubt whether such a procedure adds much to our understanding.

79 It appears that a copy of a Chinese translation of the Elements was present in the imperial library at the end of the thirteenth century, but was ignored; see Needham, 1959: 105 ff. Huff (1993: 241; with a reference to Aydin Sayili, *The Observatory in Islam*, Ankara 1960, p. 189) adds: "Even more tantalizing are the reports that a Mongol ruler in China, Mangu (d. 1257 ...) is said to 'have mastered difficult passages of Euclid by himself'."

80 Engelfriet, 1998: 297–298.

81 Martzloff, 1997: 275. Yabuuti (2000: 40) suggests that Chinese mathematicians deduced theorems like that of Pythagoras by analogy, intuitively.

82 Martzloff, 1997: 72.

83 Note further Martzloff, 1997: 276: "certain texts by Liu Hui [(end of third century C.E.)] and other mathematicians contain reasoning which, while it is not Euclidean, is no less well constructed and completely exact. Moreover, although they are not numerous, these arguments appear all the more salient because they are without peers in other non-Euclidean mathematical traditions. But they also enable us to understand that Chinese mathematics is in part based on a small number of heuristic operational methods of a geometrical type. ... In fact, the most striking thing is the concrete appearance of the Chinese approach or rather the fact that abstract results are accessed via 'concrete' means. Chinese proofs tend to be based on visual or manual illustration of certain relationships rather than on purely discursive logic."

84 Bronkhorst, 1999.

85 Cp. Collins, 1998: 551: "Organizationally, the mathematicians, astronomers, and medical doctors were based in private familistic lineages and guilds, never part of the sustained argument provided by philosophical networks. Public networks of argument did exist in India; its philosophical lineages reached high levels of abstract development. Only mathematics and science were not carried along with it." The striking absence of a Buddhist contribution to and participation in the development of astronomy and mathematics in classical India may be partly responsible for the relative "peace" enjoyed by these branches of learning. (Vogel, 1997, shows that the Buddhists – or at least some of them – were not averse to following developments in astronomy to fix the dates of their Poṣadha ceremony. The need to fix these dates did not apparently have the same effect as the Christian need to fix the date of Easter; about this latter need Duncan, 1998: 79 states: "The history of science in the Middle Ages would have been very different if the bishops at Nicaea had decided to name a fixed date for Easter in the solar calendar"; further details in Heilbron, 1999.) Note that *jyotiṣa/jyotis* is mentioned in some lists of *kalās* occurring in Buddhist texts; see Franco, 2000: 550 ([61]) with note 56.

86 A whole chapter of Brahmagupta's Brāhmasphuṭasiddhānta (no. 11: Tantraparīkṣā) is dedicated to the refutation of different opinions.

87 For a recent example (from Jainism) of the close link between calendrical (i.e. astronomical) and sectarian concerns, see Cort, 1999.

88 He calls his teacher *sarvatantrasvatantra* and *sarveṣu tantreṣu samaṃ svatantraḥ*, his father *advaitavidyācārya* (CESS 5 (1994), p. 735).

89 The one quotation from the Vākyapadīya (p. 22) concerns the meaning and function of *upāya*. Be it noted that Bhāskara's commentary on Gītikāpāda 1 enumerates the five *buddhīndriyas*, the five *karmendriyas*, plus *manas*, *buddhi* and *ahaṅkāra*, all of them known from Sāṃkhya (p. 4 ll. 11–15). Bhāskara also shows some acquaintance with Mīmāṃsā: p. 182 l. 8 *sarvaśākhāpratyayam ekaṃ karma* may have been cited from Śabara's Bhāṣya on Mīmāṃsāsūtra 2.4.9; the discussion immediately preceding this has a parallel in Śabara on 1.3.2.

90 Mathematics is most often presented in treatises of astronomy, and it seems likely that astronomers often earned their living as astrologers. Cp. Al-Bīrūnī (E.C. Sachau's translation as reproduced in Chattopadhyaya, 1992: 510): "If a man wants to gain the title of an astronomer, he must not only know scientific or mathematical astronomy, but also astrology." Cp. Pingree, 1981: 56 as cited in Yano, 1987: 54: "There was never in India a *jāti* [caste, MY] of mathematicians, and rarely anything that could be called a school; most mathematicians were *jyotiṣīs* (astronomers or astrologers)." Pingree (1993: 77) argues that Āryabhaṭa, far from making observations himself, derived the longitudes of the planets "from astrological playing with numbers".

91 On the low esteem in which astrologers were held, see Kane, 1974: 526 ff.

92 Āryabhaṭīya Bhāṣya p. 48 l. 16 ff. The edition of K.S. Shukla contains a somewhat different diagram; however, a manuscript page reproduced in Keller's thesis (I p. 223) appears to support her construction.

93 Cp. Srinivas, 1990: 35; Sarasvati Amma, 1999: 133 ff. Keller refers in this connection to an unpublished Ph.D. thesis of Simon Fraser University: A critical edition, English translation and commentary of the Upodghāta Ṣaḍvidhaprakaraṇa and Kuṭṭakādhikāra of the Sūryaprakāśa of Sūryadāsa, by Pushpa Kumari Jain, 1995.

94 David Pingree (private communication) points out that certain theorems on cyclic quadrilaterals presented by Brahmagupta (628 C.E.) were not developed from a Euclidean approach until the 16th and 17th centuries in Europe.

REFERENCES

Ascher, Marcia (1991). *Ethnomathematics*. Chapman & Hall. (I have only had access to the French translation: Ascher, 1998.)

Ascher, Marcia (1998). *Mathématiques d'ailleurs. Nombres, formes et jeux dans les sociétés traditionnelles*. Traduction de l'anglais (États-Unis) et postface de Karine Chemla et Serge Pahaut. Paris: Éditions du Seuil. (French translation of Ascher, 1991.)

Ascher, M., and R. Ascher (1994). 'Ethnomathematics' = Grattan-Guinness, 1994: II: 1545–1554.

Billard, Roger (1971). *L'astronomie indienne. Investigation des textes sanskrits et des données numériques*. Paris: Adrien-Maisonneuve. (PEFEO, 83.)

Bronkhorst, Johannes (1990). 'Vārttika', *WZKS* 34: 123–146.

Bronkhorst, Johannes (1999). *Why is There Philosophy in India?* Amsterdam: Royal Netherlands Academy of Arts and Sciences. 1999. (Sixth Gonda lecture, held on 13 November 1998 on the premises of the Royal Netherlands Academy of Arts and Sciences.)

Bronkhorst, Johannes (forthcoming). 'The Relationship Between Linguistics and Other Sciences in India', *History of the Language Sciences*. Ed. Kees Versteegh.

Bronkhorst, Johannes (forthcoming2). 'The Cāndra-vyākaraṇa: Some Questions', *Festschrift Cardona*.

Chattopadhyaya, Debiprasad (ed.) (1992). *Studies in the History of Science in India*. Vol. 2. New Delhi: Asha Jyoti.

Chemla, Karine (1997). 'What is at Stake in Mathematical Proofs from Third-century China?' *Science in Context* 10(2): 227–251.

Chemla, Karine (1999). 'Philosophical Reflections in Chinese Ancient Mathematical Texts: Liu Hui's Reference to the Yijing', in Sik Kim Yung and Francesca Bray (eds.), *Current Perspectives in the History of Science in East Asia*, pp. 89–100. Seoul: Seoul National University Press.

Clark, Walter Eugene (1930). *The Āryabhaṭīya of Āryabhaṭa. An Ancient Indian Work on Mathematics and Astronomy. Translated with Notes*. Chicago: The University of Chicago Press.

Colebrooke, Henry Thomas (1817). *Algebra, with Arithmetic and Mensuration, from the Sanscrit of Brahmegupta and Bháscara*. London: John Murray.

Collins, Randall (1998). *The Sociology of Philosophies. A Global Theory of Intellectual Change*. Cambridge, Massachusetts, and London, England: The Belknap Press of Harvard University Press.

Cort, John E. (1999). 'Fistfights in the Monastery: Calendars, Conflict and Karma Among the Jains', in N.K. Wagle and Olle Qvarnström (eds.), *Approaches to Jaina Studies: Philosophy, Logic, Rituals and Symbols*, pp. 36–59. Toronto: Centre for South Asian Studies, University of Toronto. (South Asian Studies Papers, no. 11.)

Datta, B. B. (1929). 'The Mathematical Achievements of the Jainas', *Bulletin of the Calcutta Mathematical Society*: 115–145. Reprinted in Chattopadhyaya, 1992: II: 684–716 (references are to the reprint).

Davis, Philip J. and Reuben Hersh (1981). *The Mathematical Experience*. Boston etc.: Birkhäuser.

Duncan, David Ewing (1998). *The Calendar. The 5000-year Struggle to Align the Clock and the Heavens – and What Happened to the Missing Ten Days*. Reprint: London: Fourth Estate, 1999.

Elfering, Kurt (1968). 'Über den Flächen bzw. Rauminhalt von Dreieck und Pyramide sowie Kreis und Kugel bei Āryabhaṭa I', *Rechenpfenninge. Festschrift für Kurt Vogel*, pp. 57–67. München (not seen).

Elfering, Kurt (1975). *Die Mathematik des Āryabhaṭa I. Text, Übersetzung aus dem Sanskrit und Kommentar*. München: Wilhelm Fink.

Elfering, Kurt (1977). 'The Area of a Triangle and the Volume of a Pyramid as Well as the Area of a Circle and the Surface of the Hemisphere in the Mathematics of Āryabhaṭa I', *Indian Journal of the History of Science* 12: 232–236 (not seen).

Engelfriet, Peter M. (1998). *Euclid in China. The Genesis of the First Chinese Translation of Euclid's Elements Books I–VI (Jihe Yuanben; Beijing, 1607) and its Reception up to 1723*. Leiden etc.: Brill. (Sinica Leidensia, 40.)

Filliozat, Pierre-Sylvain (1988). 'Calculs de demi-cordes d'arcs par Āryabhaṭa et Bhāskara I', BEI 6: 255–274.

Filliozat, Pierre-Sylvain (1995). 'Sanskrit Linguistics and Mathematics in Ancient India', *Indian Horizons* 44(4): 39–50.

Filliozat, Pierre-Sylvain, and Guy Mazars (1985). 'Observations sur la formule du volume de la pyramide et de la sphère chez Āryabhaṭa', BEI 3, 37–48.

Franco, Eli (2000). 'The Spitzer Manuscript – report on Work in Progress', edited by the Committee for the Felicitation of Professor Doctor Junsho Kato's Sixtieth Birthday, *Abhidharma and Indian Thought. Essays in Honor of Professor Doctor Junsho Kato on his Sixtieth Birthday*, pp. 562–544. Nagoya. Tokyo: Shunju-sha. (= [49]–[67]).

Friberg, J. (1990). 'Mathematik', *Reallexikon der Assyriologie und vorderasiatischen Archäologie* VII.7/8: 531–585.

Gillings, Richard J. (1972). *Mathematics in the Time of the Pharaohs*. Cambridge, Massachusettes & London, England: MIT Press.

Grattan-Guinness, I. (ed.) (1994). *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*. 2 vols. London and New York: Routledge.

Hayashi, Takao (1994). 'Indian Mathematics', = Grattan-Guinness, 1994: I: 118–130.

Hayashi, Takao (1995). *The Bhakshālī Manuscript. An Ancient Indian Mathematical Treatise*. Groningen: Egbert Forsten. (Groningen Oriental Studies, 11.)

Hayashi, Takao (1997). 'Calculations of the Surface of a Sphere in India', *The Science and Engineering Review of Doshisha University* 37(4): 194–238.

Heath, T. (1921). *History of Greek Mathematics*. 2 vols. Reprint: New York: Dover Publications, 1981.

Heilbron, J.L. (1999). *The Sun in the Church. Cathedrals as Solar Observatories*. Cambridge, Mass. & London, England: Harvard University Press.

Huff, Toby E. (1993). *The Rise of Early Modern Science: Islam, China and the West*. Cambridge etc.: Cambridge University Press.

Ingalls, Daniel H.H. (1954). 'The Comparison of Indian and Western Philosophy', JORM 22: 1–11.

Joseph, George Gheverghese (1990). *The Crest of the Peacock. Non-European Roots of Mathematics*. London – New York: I.B. Tauris.

Kane, Pandurang Vaman (1974). *History of Dharmaśāstra (Ancient and Mediaeval Religious and Civil Law)*. Vol. V, Part I. Poona: Bhandarkar Oriental Research Institute. (Government Oriental Series, Class B, No. 6.)

Kaye, G. R. (1919). 'Influence grecque dans le développement des mathématiques hindoues', *Scientia* 25: 1–14.

Keller, Agathe (2000). *Un commentaire indien du VIIème siècle. Bhāskara et le gaṇita-pāda de l'Āryabhaṭīya*. Thèse de doctorat. Université Paris 7. Soutenue le 21 mars 2000.

Kline, Morris (1972). *Mathematical Thought from Ancient to Modern Times*. Volume I. New York – Oxford: Oxford University Press.

Kuppuram, G. and K. Kumudamani (1990). *History of Science and Technology in India. Vol. II: Mathematics, Astronomy*. Delhi: Sundeep Prakashan.

Lindtner, Christian (2000). 'King Jagajjyotirmalla's Ślokasārasaṃgrahaḥ', AS 54(1): 47–66.

Lloyd, G.E.R. (1990). *Demystifying Mentalities*. Cambridge etc.: Cambridge University Press. Reprinted 1993.

Martzloff, Jean-Claude (1997). *A History of Chinese Mathematics* (Translation of *Histoire des mathématiques chinoises*, Masson, Paris 1987, tr. Stephen S. Wilson). Berlin: Springer.

Meulenbeld, G.J. (1999 ff.). *A History of Indian Medical Literature*. To be completed in 3 vols. Groningen: Egbert Forsten. (Groningen Oriental Studies, 15.)

Michaels, Axel (1978). *Beweisverfahren in der vedischen Sakralgeometrie: ein Beitrag zur Entstehungsgeschichte von Wissenschaft*. Wiesbaden: Franz Steiner. (Alt- und Neu-Indische Studien, 20.)

Müller, Conrad (1940). 'Volumen und Oberfläche der Kugel bei Āryabhaṭa I', *Deutsche Mathematik* 5(3): 244–255.

Needham, Joseph (1959). *Science and Civilisation in China. Vol. 3: Mathematics and the Sciences of the Heavens and the Earth*. Cambridge: Cambridge University Press.

Netz, Reviel (1999). *The Shaping of Deduction in Greek Mathematics. A Study in Cognitive History*. Cambridge etc.: Cambridge University Press.

Neugebauer, O. (1957). *The Exact Sciences in Antiquity*. 2nd ed. Providence, Rhode Island: Brown University Press.

Pādalipta: *Jyotiṣkaraṇḍaka*. In: *Paiṇṇayasuttāiṃ Part III: Pādaliptasūri's Joïsakaraṇḍagaṃ with Prakṛta Ṭippanaka by Vācaka Śivanandi*. Ed. Muni Shri Puṇyavijayaji. Bombay: Shri Mahāvīra Jaina Vidyālaya. (Jaina-Āgama-Series no. 17, part III.)

Pingree, David (1970–1994). *Census of the Exact Sciences in Sanskrit*. Series A, vol. 1–5. Philadelphia: American Philosophical Society.

Pingree, David (1971). 'On the Greek Origin of the Indian Planetary Model Employing a Double Epicycle', *Journal for the History of Astronomy* 2: 80–85.

Pingree, David (1978). 'History of Mathematical Astronomy in India', in Charles Coulston Gillispie (ed.), *Dictionary of Scientific Biography*. Vol. XV, Supplement I, pp. 533–633. New York: Charles Scribner's Sons.

Pingree, David (1981). *Jyotiḥśāstra. Astral and Mathematical Literature*. Wiesbaden: Otto Harrassowitz. (HIL vol. VI, fasc. 4.)

Pingree, David (1993). 'Aryabhaṭa, the Paitāmahasiddhānta, and Greek Astronomy', *Studies in History of Medicine and Science*, new series 12(1–2): 69–79.

Plofker, Kim (1996). 'How to Appreciate Indian Techniques for Deriving Mathematical Formulas?' in Catherine Goldstein, Jeremy Gray and Jim Ritter (eds.), *L'Europe mathématique: histoires, mythes, identités/Mathematical Europe: History, Myth, Identity*, pp. 53–65. Paris: Éditions de la Maison des sciences de l'homme.

Pottage, John (1974). 'The Mensuration of Quadrilaterals and the Generation of Pythagorean Triads: A Mathematical, Heuristical and Historical Study with Special Reference to Brahmagupta's Rules', AHES 12, 299–354.

Prakash Sarasvati, Svami Satya (1986). *A Critical Study of Brahmagupta and his Works (the Most Distinguished Indian Astronomer and Mathematician of the Sixth Century A.D.)*. Delhi: Govindram Hasanand.

Ruegg, David Seyfort (1978). 'Mathematical and Linguistic Models in Indian Thought: The Case of Śūnyatā', WZKS 22: 171–181.

Sarasvati Amma, T.A. (1999). *Geometry in Ancient and Medieval India*. Second revised edition. Delhi: Motilal Banarsidass.

Schroeder, L. v. (1884). *Pythagoras und die Inder. Eine Untersuchung über Herkunft und Abstammung der pythagoreischen Lehren*. Leipzig: Otto Schulze.

Seidenberg, A. (1962). 'The Ritual Origin of Geometry', AHES 1: 488–527.

Seidenberg, A. (1975). 'Did Euclid's Elements, Book I, Develop Geometry Axiomatically?' AHES 14: 263–295.

Seidenberg, A. (1978). 'The Origin of Mathematics', AHES 18: 301–342.

Seidenberg, A. (1983). 'The Geometry of the Vedic Rituals', = Staal, 1983: II: 95–126.

Shapiro, Stewart (1997). *Philosophy of Mathematics. Structure and Ontology*. New York – Oxford: Oxford University Press.

Sharma, Ram Swarup (1966). *Shri Brahmagupta viracita Brāhma-sphuṭa Siddhānta, with Vāsanā, Vijñāna and Hindi Commentaries*. Edited by a board of editors headed by ... Ram Swarup Sharma. 4 vols. New Delhi: Indian Institute of Astronomical and Sanskrit Research.

Shukla, Kripa Shankar (1972). 'Hindu Mathematics in the Seventh Century as Found in Bhāskara I's Commentary on the Āryabhaṭīya (IV)', *Gaṇita* 23(2): 41–50.

Shukla, Kripa Shankar (ed.) (1976). *Āryabhaṭīya of Āryabhaṭa, with the Commentary of Bhāskara I and Someśvara*. New Delhi: Indian National Science Academy. (Āryabhaṭīya Critical Edition Series, Pt. 2.)

Singh, Navjyoti (1990). 'Foundations of Logic in Ancient India: Linguistics and Mathematics', = Kuppuram and Kumudamani, 1990: 245–287.

Smeur, A.J.E.M. (1970). 'On the Value Equivalent to π in Ancient Mathematical Texts: A New Interpretation', AHES 6: 249–270.

Smith, David Eugene (1923). *History of Mathematics, Vol. I: General Survey of the History of Elementary Mathematics*. Reprint: Dover, New York, 1958.

Srinivas, M.D. (1990). 'The Methodology of Indian Mathematics and its Contemporary Relevance', = Kuppuram and Kumudamani, 1990: 29–86.

Staal, Frits (1960). Review of *Contributions à l'histoire de la philosophie linguistique indienne* by D.S. Ruegg. PEW 10: 53–57.

Staal, Frits (1963). 'Euclides and Pāṇini: twee methodische richtlijnen voor de filosofie', (Amsterdam: Polak & Van Gennep) = Staal, 1986: 77–115.

Staal, Frits (1983). *AGNI. The Vedic Ritual of the Fire Altar*. 2 vols. Berkeley: Asian Humanities Press.

Staal, Frits (1986). *Over zin en onzin in filosofie, religie en wetenschap*. Amsterdam: Meulenhoff Informatief.

Staal, Frits (1988). *Universals. Studies in Indian Logic and Linguistics*. University of Chicago Press.

Staal, Frits (1993). *Concepts of Science in Europe and Asia*. Leiden: International Institute for Asian Studies.

Staal, Frits (1999). 'Greek and Vedic Geometry', JIP 27(1/2) (Guruvandana: Essays in Indology in Honour of K. Bhattacharya), 105–127.

Staal, J.F. (= Frits) (1965). 'Euclid and Pāṇini' (PEW 15, 99–116) = Staal, 1988: 143–160.

Subbarayappa, B.V., and K.V. Sarma (1985). *Indian Astronomy: A Source Book (Based Primarily on Sanskrit Texts)*. Bombay: Nehru Centre.

Umāsvāti (1926, 1930). *Tattvārthādhigama Sūtra Bhāṣya*. Edited, with the commentaries of Devaguptasūri and Siddhasenagaṇi, by Hiralal Rasikdas Kapadia. 2 vols. Bombay: Jivanchand Sakerchand Javeri.

van der Waerden, B.L. (1980). 'On Pre-Babylonian Mathematics, I, and II', AHES 23: 1–25 & 27–46.

van der Waerden, Bartel Leendert (1983). *Geometry and Algebra in Ancient Civilizations*. Berlin etc.: Springer.

Vogel, Claus (1997). 'On the Date of the Poṣadha Ceremony as Taught by the Mūlasarvāstivādins', in Petra Kieffer-Pülz and Jens-Uwe Hartmann (eds.), *Bauddhavidyāsudhākaraḥ. Studies in Honour of Heinz Bechert on the Occasion of His 65th Birthday*, pp. 673–688. Swisttal-Odendorf: Indica et Tibetica Verlag. (IndTib 30.)

Wujastyk, Dominik (1993). *Metarules of Pāṇinian Grammar. Vyāḍi's Paribhāṣāvṛtti*. 2 vols. Groningen: Egbert Forsten. (Groningen Oriental Studies, 5)

Yabuuti, Kiyosi (2000). *Une histoire des mathématiques chinoises*. Traduit du japonais par Kaoru Baba et Catherine Jami. Paris: Belin – Pour La Science.
Yano, Michio (1987). 'Science and Religion in Ancient India', *Zen Buddhism Today* 5: 50–59.

ABBREVIATIONS

AHES	Archive for History of Exact Sciences
AS	Asiatische Studien, Études Asiatiques, Bern
BEI	Bulletin d'Études Indiennes, Paris
CESS	David Pingree, Census of the Exact Sciences in Sanskrit, Series A, vol. 1–5, Philadelphia 1970–1994
HIL	A History of Indian Literature, ed. J. Gonda, Wiesbaden 1973 ff.
IndTib	Indica et Tibetica, Bonn
JA	Journal Asiatique, Paris
JIP	Journal of Indian Philosophy
JORM	Journal of Oriental Research, Madras
Mahā-bh	Patañjali (Vyākaraṇa-)Mahābhāṣya, ed. F. Kielhorn, Bombay 1880–1885
Pāṇ	Pāṇinian sūtra
PEFEO	Publications de l'École Française d'Extrême-Orient, Paris
PEW	Philosophy East and West, Hawaii
Vkp	Bhartṛhari, Vākyapadīya, ed. W. Rau, Wiesbaden 1977
vt.	vārttika on Pāṇinian sūtra
WZKS	Wiener Zeitschrift für die Kunde Südasiens, Wien

Section de langues et civilisations orientales
Université de Lausanne
CH-1015 Lausanne
Switzerland